Symmetries, Groups

1 Symmetries

There are a lot of connections between Platonic solids (and, more generally, polytopes) and other topics in math. One such topic is *groups*, in particular, symmetry groups. We will define an abstract group later, after looking at a number of examples.

The article for Quanta called "Symmetry, algebra and the monster" by Patrick Honner gives a quick introduction to symmetry groups by examining the symmetries of the square. Incidentally, his article is meant to give some background regarding recent research on a group called the "monster group", discussed in some other articles he links, which is why there is a reference to a monster in the title, but no monsters in his article.

Formally, a *symmetry* of the square is a transformation of the square, which we can think of as a function mapping from the square to itself that is an *isometry*, meaning that it is distance-preserving. As we will see, any symmetry of the square will be a mirror-type symmetry, or a rotational symmetry, or a combination of those.

2 Symmetries of squares and other polygons

To begin, we'll take a look at the symmetries of the square.

1. To keep track of all the symmetries, it helps to label the corners of the square. Why are these labels enough to distinguish all distinct symmetries?
2. By making the symmetries one after the other, we can make new symmetries. However, we find that there are many relationships between different symmetries.
   (a) Consider the symmetry that you get from rotating the square 90° clockwise and then mirroring it across a vertical axis. This symmetry corresponds to a reflection. Which one?
   (b) Why does a rotation followed by a reflection always give a reflection?
   (c) Why does a reflection followed by another reflection give a rotation?
3. Consider the symmetry that you get from rotating the square 90° clockwise and then mirroring it across a vertical axis. Compare it to the symmetry that you get from mirroring the square across a vertical axis and then rotating it 90° clockwise. Are they the same or different?
4. How many symmetries does a square have?
5. How many symmetries does an equilateral triangle have?
6. How many symmetries does a regular pentagon have?
7. How many symmetries does a regular n-gon have?

3 Symmetries of Platonic solids and duals of Platonic solids

Next, we'll take a look at the symmetries of the cube.

1. To keep track of the symmetries of the cube, we can label all of the corners. Why are these labels enough to distinguish all distinct symmetries?
2. Find all the axes around which the cube has rotational symmetry. How many are there?

3. Reflective symmetries of the cube come from flipping across a plane rather than a line. Find a plane across which the cube has reflective symmetry. What does reflecting across this plane do to the labels on the vertices of the cube? Is it possible to move the cube in a rigid way between these two positions in the same way that it's possible to flip a square between its reflections?

4. To make things possibly a little easier on ourselves, we could try labeling the faces of the cube rather than the vertices since the cube has fewer faces than vertices. Why is labeling the faces enough to distinguish all the symmetries?

5. How many total symmetries does the cube have? You may find the previous question to be helpful.

   Next, we will find the number of symmetries that an octahedron has. However, rather than counting them directly, we will find them indirectly using the idea of the *dual* of a given Platonic solid.

6. Recall the idea of the dual of a Platonic solid that we introduced at the end of the day yesterday: Which Platonic solids were dual to one another? Why, in general, is the dual of a Platonic solid itself a Platonic solid? Why does taking the dual of a Platonic solid twice get you back the original Platonic solid?

7. Why does the dual of a Platonic solid have the same number of symmetries as the original one? Do rotations of the original Platonic solid correspond to rotations or reflections of the dual?

8. Given the answer to the previous question, how many symmetries does the octahedron have?

9. Find how many symmetries the other Platonic solids have.

4 Groups

The symmetries of the polytopes in the last section are examples of *groups*. More specifically in those cases they're called *symmetry groups*. The symmetry group of a polygon is called a *dihedral group*. The symmetry group of a cube is called the *octahedral group*.

In class, we will talk about the definition of a group.

1. Consider the set of integers with the operation of addition. Does this form a group? This happens to be a group where the order in which we compose two elements of the group doesn't matter. Groups like this are called *abelian groups*.

2. Consider the set of integers with the operation of multiplication. Why doesn't it form a group? Do we get a group if we get rid of 0?

3. Consider the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Do they form a group with the operation given by composition of functions? What would the identity function be? How would you take inverses?

4. How many different possible orders are there in which we can write the numbers $1, 2, 3, 4$? This is the same as asking how many possible permutations there are of four objects. We can think of these permutations as functions that are one-to-one and onto from the set $\{1, 2, 3, 4\}$ to itself, and we can think of composing permutations in terms of composing these functions.

   (a) What permutation do we get if we perform the permutation that switches 1 and 2 followed by the permutation that switches 3 and 4?

   (b) What permutation do we get if we perform the permutation that switches 1 and 3 followed by the permutation that switches 3 and 4?

   (c) The set of permutations of four objects with this composition operation is a group. What is the identity of this group? How do you take inverses?

   (d) For any positive integer n, we can think about the permutation group of n objects. How many elements does it have?

5 Subgroups

Sometimes groups are sitting inside other large groups where they share the same operation and identity. The smaller groups are called *subgroups* of the larger groups they sit inside.

1. How can we think of the group of symmetries of the square as a subgroup of the group of permutations of four objects? (Hint: This was discussed in Patrick Honner's article)

The *order* of a group is the number of elements it has in it. One of the reasons that thinking about subgroups of larger groups is helpful is the following fact: the order of the subgroup divides the order of the group it's sitting inside.

2. Verify that the order of the group of the symmetries of the square divides the order of the group of permutations of four objects.
3. All of the symmetry groups we looked at in previous sections are subgroups of permutation groups. Which symmetry groups are subgroups of which permutation groups? There will be more than one answer.
4. In fact each symmetry group is sitting inside infinitely many permutation groups because the group of permutations of n objects is a subgroup of the group of permutations of m objects whenever $n \leq m$. Can you find an argument for why this might be true? It might be easier to start by thinking about the group of permutations of n objects sitting inside the group of permutations of $n+1$ objects.

6 Cyclic groups

1. Consider the element of the symmetry group of the square given by clockwise rotation by 90°. What happens when we compose this symmetry with itself repeatedly? Why does the set of all the symmetries we get this way form a subgroup of the symmetry group of the square?

We say that the subgroup we got in the last problem is *generated* by the symmetry given by rotation by 90°.

2. In fact, given a group G with a finite number of elements and any element g, the set of elements we get from composing g with itself repeatedly will be a subgroup of G. Why is this true?
3. Consider the other elements of the symmetry group of the square. What subgroups do they generate? Which elements generate the same subgroups?
4. Cyclic subgroups (that is, groups generated by one element) are always abelian. Can you find a justification for why this must be true?

7 More...

1. We don't get to use our fun fact about order here since the integers have infinte order, but it's still pretty fun to find all the subgroups of the group of integers with the addition operation.
2. Now that we've worked through some material related to Patrick Honner's article in class, take another read through the article. Are there any aspects of the article that have become more clear, or are still unclear? What thoughts or reactions do you have about the article?