Math 817, Fall 2023, Dr. Honigs
Homework 1, Part 1 (of two parts)
Homework 1 (all of it) is due Tues. **Sept. 19** at the start of class

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Instructions:

- You are encouraged to work in groups. However, write your solutions in your own words.
- In your solutions, list any names of people you have worked with. (This is practice for listing collaborators!)
- In your solutions, refrain from using language like "clear", "obvious", or "easy".

Homework 1 Part 1 will be graded out of 35 points.

Textbook questions:

Aluffi Ch. I

- Exercises 5.4-5.6 (Categories and universal properties)

Notes: In these exercises, don't just give the answers, also prove they are correct.
5.6: Consider the product and coproduct of two arbitrary elements (as opposed to, say, that of infinitely elements).
Each of the textbook questions here is worth 5 points.

Additional questions:

1. (10 points) (Monomorphisms and Epimorphisms) The structure of a category just has information about its objects and morphisms and compositions of the morphisms. In this exercise we will consider injectivity (one-to-one) and surjectivity (onto) along with categorical analogues that a category "knows" about.

   (a) Let S and T be nonempty sets and $f : S \to T$ a function. Consider the following statements. One statement is equivalent to f being injective and the other is equivalent to f being surjective. Decide which one is which and prove the equivalences.

      (i) f has a left inverse, i.e. there is a function $g : T \to S$ so that $g \circ f = 1_S$.

      (ii) f has a right inverse, i.e. there is a function $h : T \to S$ so that $f \circ h = 1_T$.

   (b) Let $\mathcal{C}$ be a category. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is a *monomorphism* if for any $C \in \mathrm{Obj}(\mathcal{C})$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(C, A)$,

   $$f \circ \alpha = f \circ \beta \Rightarrow \alpha = \beta.$$

   A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an *epimorphism* if for any $C \in \mathrm{Obj}(\mathcal{C})$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(B, C)$,

   $$\alpha \circ f = \beta \circ f \Rightarrow \alpha = \beta.$$

   Show that if f

(i) ...has a left inverse then it is a monomorphism.

(ii) ...has a right inverse then it is an epimorphism.

(c) Prove that in the category Set, a morphism is monic if and only if it is injective and epic if and only if it is surjective.

(d) Let S be a set and consider the category of subsets $\hat{S}$ we defined in lecture. Which of the morphisms of $\hat{S}$ are monomorphisms? epimorphisms? isomorphisms?

(Note: It's a consequence of this exercise that morphisms may be both monic and epic but not isomorphisms.)

(e) Consider the morphism $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in the category Ring.

(i) We will assume that ring homomorphisms send the multiplicative identity to the multiplicative identity. Show that this assumption means there is a unique morphism $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

(ii) Show that the morphism $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in Ring. (Note that this map is epic but not surjective!)

2. (10 points) (Categories made with relations) There is a category Int whose objects are the integers $\mathbb{Z}$ and

$$\mathrm{Hom}_{\mathsf{Int}}(m,n) = \begin{cases} \{*\} & \text{if } m \le n \\ \emptyset & \text{otherwise} \end{cases}$$

(a) Explain why, if we replace $\le$ with $<$ in the definition of Int, we *don't* get a category.

(b) Let S be a set with a binary relation $\sim$. We can generalize the construction in the previous parts by trying to construct a category whose objects are the elements of S and where for any $s, t \in S$, the set of morphisms from s to t has one element if $s \sim t$ and is empty otherwise.

If S is an *equivalence relation*, it is *symmmetric, reflexive* and *transitive*. Which of these three qualities are necessary for the above construction to give a category? Prove your answer.

(c) A category is called a *groupoid* if all its morphisms are isomorphisms. Prove that the construction from part (b) gives a groupoid if and only if $\sim$ is an equivalence relation.

(d) Consider the slice and coslice categories Int_3 and Int^3 where Int is the category defined in 2(a). Describe the objects and morphisms of these categories.