Math 817, Fall 2023, Dr. Honigs
Homework 5, Part 1
Due **Nov. 14** at the start of class

Instructions:

- You are encouraged to work in groups. However, write your solutions in your own words.
- In your solutions, list any names of people you have worked with. (This is practice for listing collaborators!)
- In your solutions, refrain from using language like "clear", "obvious", or "easy".

Homework 5 Part 1 will be graded out of 50 points. Each problem is worth 5 points.

Textbook questions:

Aluffi Ch. IV

- 5.13, 5.14 (more practice with semidirect products)

Aluffi Ch. III

- 5.4, 5.10, 5.11, 6.14, 6.18 (practice with rings, modules, finite generation)

Aluffi Ch. VI

- 4.3, 4.5 (also practice with rings, modules, finite generation)

Additional question:

1. Let $1 \to N \xrightarrow{\varphi} G \xrightarrow{\psi} H \to 1$ be a short exact sequence of groups that is split exact, that is, there is a group homomorphism $\iota : H \to G$ so that $\psi \circ \iota = \mathrm{id}_H$.
   (a) Exhibit an isomorphism $\beta : N \rtimes_\theta H \to G$. (This will be the map we used in class. It must be stated what θ is in order to define it.) Verify that β is an isomorphism.
   (b) Let R be a ring. Suppose that our situation is in the category $\mathsf{R\text{-}Mod}$ rather than the category Grp, i.e., all the objects are R-modules rather than groups and all the homomorphisms are R-module maps. Show that $G \cong N \times H$. (This doesn't need to be long, you may reference your answer to part (a).)

Remarks:

- In IV 5.13, the action of G on G/H is the action by left multiplication. The "kernel" of an action is an element in the group that acts identically on every element in the set being acted on, i.e. is in the intersection of all stabilizers.
- IV 5.14 has the (possibly somewhat oblique?) hint to use 5.13. This hint seems meant to help you guess how to correctly think of $C_2 \times C_2$ and S_3 as subgroups of S_4. You may find other ways to do this, but I recommend picking an injection of S_3 into S_4 first and going from there.