Math 817, Fall 2023, Dr. Honigs
Homework 6, Part 2
Due ~~Nov. 28~~ **Nov. 30** at the start of class

Instructions:

- You are encouraged to work in groups. However, write your solutions in your own words.
- In your solutions, list any names of people you have worked with. (This is practice for listing collaborators!)
- In your solutions, refrain from using language like "clear", "obvious", or "easy".

Homework 6 Part 2 will be graded out of 50 points. Questions 2, 5, and 6 are worth 10 points. The other questions are worth 5 points each.

Remark: These questions are designed to be done *without* using characters.
Questions:

1. Draw an equilateral triangle in the plane with vertices at $(1,0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. In this question you will use the geometry of this triangle to consider a representations.
   (a) In the standard basis, give a matrix for the element of D_6 rotating the triangle clockwise. Then, give a matrix the element of D_6 that reflects the triangle across the x-axis.
   (b) Find a new basis that diagonalizes the first matrix. Then write down the second matrix in this new basis

2. (a) Show that any degree 1 representation of a group G must be constant over conjugacy classes (i.e., conjugate elements of G must map to the same element).
   (b) Recall that S_n is generated by transpositions, and all transpositions are conjugate to one another. Prove that S_n has exactly two degree 1 representations.

3. Let $G = D_8 = \langle \tau, \sigma \mid \tau^4 = \sigma^2 = e, \sigma\tau\sigma = \tau^{-1} \rangle$. There is a 3-dimensional representation $\rho : G \to \mathrm{GL}_3(\mathbb{C})$ defined by

$$\rho_\tau = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \rho_\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

   Find a 1-dimensional G-invariant subspace of $\mathbb{C}^3$. Is it possible to find another one? Deduce that ρ can be decomposed as a direct sum of representations $\mathbb{C}^3 = U_1 \oplus U_2$ where U_2 is degree 2.

4. How many irreducible representations of C_6 are there? How many of these are faithful? (A *faithful* representation is one where the group homomorphism $\rho : G \to \mathrm{GL}(V)$ is injective.)

5. Let $G = S_3$ with generators $\tau = (123), \sigma = (12)$. Let $\rho : G \to \mathrm{GL}(V)$ be the degree 2 representation given as follows, where $\zeta = e^{\frac{2\pi i}{3}}$:

$$\rho_\tau = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \rho_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   (a) The set of linear transformations $\mathrm{Hom}_{\mathbb{C}}(V, V)$ has a vector space structure. What is its dimension? Given a basis $\{e_i\}_{i=1}^n$ for V, give a natural basis for $\mathrm{Hom}_{\mathbb{C}}(V, V)$.

   (b) There is a natural representation that ρ induces on $\mathrm{Hom}_{\mathbb{C}}(V, V)$. Write it down in terms of the basis you described in the previous part.

   (c) How do we know that the G-invariant subspace $\mathrm{Hom}_{\mathbb{C}}^G(V, V)$ must be 1-dimensional? Find a vector that spans it.

   (d) Find the decomposition of $\mathrm{Hom}_{\mathbb{C}}(V, V)$ into irreducible representations.

6. Let $G = D_{2k} = \langle \tau, \sigma \mid \tau^k = \sigma^2 = e, \sigma\tau\sigma = \tau^{-1} \rangle$. Let $V = \mathbb{C}^2$ and $\rho : G \to \mathrm{GL}(V)$ the following representation, where $\zeta = e^{\frac{2\pi i}{k}}$ is a k-th root of unity.

$$\rho_\tau = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \rho_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Let $W = \mathbb{C}$ and $\nu : G \to \mathrm{GL}(W)$ be the representation where $\nu_\tau = \mathrm{id}_W$, $\nu_\sigma = -1 \cdot \mathrm{id}_W$

   (a) Verify that ρ is a representation.

   (b) Using the standard bases, write down the natural representations V^*, $V \otimes W$, $\mathrm{Hom}(V, W)$.

7. Let $G = D_{10} = \langle \tau, \sigma \mid \tau^5 = \sigma^2 = e, \sigma\tau\sigma = \tau^{-1} \rangle$. (You may wish to use the decomposition of regular representations, to appear in class Thurs. Nov 23, to solve this problem)

   (a) Show that G has exactly two 1-dimensional representations.

   (b) Find how many irreducible representations G has, and find their degrees.