Math 818, Fall 2022, Dr. Honigs
Homework 1
Due Wed. Sept. 14 at the start of class

Instructions: You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

1. Let k be a field. Prove that there is a bijection between the following two sets:

$$\{\text{ring homomorphisms } \phi : k[x] \to k \mid \phi(a) = a \ \forall a \in k\} \Leftrightarrow \{b \mid b \in k\}$$

   That is, describe how the bijection goes and prove that it is, in fact, a bijection. (Possible hint: Start by experimenting by defining ring homomorphisms $\phi : k[x] \to k$.)

2. Let $\phi : R \to S$ be a ring homomorphism and J be an ideal of S.
   (a) Show that $\phi^{-1}(J)$ is an ideal of R.
   (b) Let I be a proper ideal of R. Show there is a bijection between ideals/prime ideals/maximal ideals of R/I and the ideals/prime ideals/maximal ideals of R that contain I. Prove this directly, without appealing to the third isomorphism theorem. You will likely find part (a) to be helpful.

3. Describe all the maximal ideals of $\mathbb{C}[x]$ (and prove that you have found them).

4. (a) Find the rational parametrization of the hyperbola $y^2 - x^2 = 1$ given by stereographic projection through the point $(0, 1)$.
   (b) Use the method shown in class and your answer to part (a) to solve the following integral:

$$\int \frac{1}{\sqrt{x^2+1}}\, dx$$

5. Let k be a field and let $k[[x]]$ be the ring of formal power series in the indeterminate x. The elements are of the following form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

   Addition and multiplication are defined by extending polynomial addition and multiplication:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$
$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

(a) Show that $1 - x$ is a unit in $k[[x]]$ with inverse $1 + x + x^2 + \cdots$.

(b) Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $k[[x]]$ if and only if $a_0 \neq 0$.

(c) Determine all of the ideals of the ring $k[[x]]$.