Math 818, Fall 2022, Dr. Honigs
Homework 2
Due Wed. Sept. 21 at the start of class

Instructions: You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

1. Show that if all the ideals in R are finitely generated, then R satisfies the ascending chain condition.

2. Show that if R satisfies the ascending chain condition, then any quotient ring R/I satisfies the asecending chain condition.

3. In this problem, you will prove the following statement: If a ring R is Noetherian and a UFD then every (nonzero) prime ideal in R is generated by finitely many irreducible elements.

   I've broken this question into parts to guide you through a solution, but if you would like a challenge, you might enjoy trying to prove the statement directly and not looking at the parts below.

   (a) Let $\mathfrak{p}$ a nonzero prime ideal in R. Let $a \in \mathfrak{p}$. Since R is a UFD, we may write a as a product of irreducible elements in R: $a = b_1 \cdots b_n$. Prove that one of these irreducible factors must be contained in $\mathfrak{p}$, call it f_1.

   (b) Now, suppose $f_1, \ldots, f_n \in \mathfrak{p}$ are irreducible elements. Show that if the ideal $(f_1, \ldots, f_n)$ is properly contained in $\mathfrak{p}$, then it is possible to produce an irreducible element in $\mathfrak{p} \setminus (f_1, \ldots, f_n)$.

   (c) Observe that parts (a) and (b) allow us to produce a sequence of ideals contained in $\mathfrak{p}$ that are generated by irreducibles:

   $$(f_1) \subsetneq (f_1, f_2) \subsetneq \cdots \subsetneq (f_1, \ldots, f_n) \subsetneq \cdots$$

   Explain why this sequence has to stabilize and finish proving the statement.

4. (a) Let I and J be ideals of a polynomial ring $k[x_1, \ldots, x_n]$. The product ideal IJ is defined to be the ideal generated by all elements of the form fg for $f \in I$ and $g \in J$. Show that $V(IJ) = V(I \cap J) = V(I) \cup V(J)$. (Hint: Reduce to needing to show that $V(IJ) \subseteq V(I) \cup V(J)$.)

   (b) Find an example of I and J where $I \cap J$ and IJ are distinct. (Possible hint: Examples where I and J are principally generated exist.)

5. Practice with fields.

   (a) Now suppose we have field extensions $L/F/k$ where $w_1, \ldots, w_m$ is a basis for L over F and $v_1, \ldots, v_n$ is a basis for F over k. What is the degree of L over k? Give a basis for L as a vector space over k and prove that it is a basis.

(b) Let L/k be a field extension and $\alpha \in L$. Consider the field $k(\alpha)$. Show that if F is a subfield of L that contains both k and α, then there is an inclusion $k(\alpha) \hookrightarrow F$.

You may use facts from lecture. This result proves in a formal way that $k(\alpha)$ is the "smallest field" containing both k and α.