

Math 818, Fall 2022, Dr. Honigs
Homework 3
Due Wed. Oct. 5 at the start of class

Instructions: You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

Reminder: Do not use terms like “clear”, “obvious”, or “easy” in your write-up.

Questions 1-3 are concerned with integral elements and extensions. Look in section 1.9 of Fulton for a reminder on the definition.

1. Let A be an integral domain that is a finitely generated R -algebra with generators a_1, \dots, a_n and suppose we have $r \in R$ so that ra_1, \dots, ra_n are integral over R . Show that for each $z \in A$, there is some $N \in \mathbb{N}$ so that $r^N z$ is integral over R .

You may use the corollary in Fulton §1.9.

2. Let $k(x)$ be a transcendental field extension over k .
 - (a) Show that any element of $k(x)$ that is integral over $k[x]$ is itself an element of $k[x]$. (There is a hint in Fulton problem 1.49(a))
 - (b) Show that there is no $a \in k[x]$ such that for any $z \in k(x)$, there is an $N \in \mathbb{N}$ so that $a^N z$ is integral over $k[x]$.
3. Suppose an integral domain S is a finitely generated R -algebra. Show that S is finitely generated as an R -module if and only if S is integral over R . You may use results from Fulton §1.9.
4. Let I be an ideal in a ring R . Prove that the radical \sqrt{I} is an ideal of R .
5. Prove that Version 1 of Weak Nullstellensatz implies Version 2.
6. Prove that Strong Nullstellensatz implies Version 1 of Weak Nullstellensatz.
7. Prove that Strong Nullstellensatz implies Version 2 of Weak Nullstellensatz. Do not simply refer to the previous two exercises, find a direct proof.