Math 818, Fall 2024, Dr. Honigs
Homework 4
Due Fri. Nov. 1

Instructions: You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

Complete the following textbook exercises and questions.

Exercises:

- In Gathmann's "Algebraic Geometry" (AG): 6.14, 6.30, 6.31, 6.36

6.31: You may use Exercise 11.33(b) from the CA notes.

Questions:

1. Let U be an open set of an *irreducible* affine variety X. Let $\varphi, \psi \in \mathcal{O}_X(U)$. Show that if there is a point $a \in U$ where the images of φ and ψ in $\mathcal{O}_{X,a}$ are equal, then $\varphi = \psi$.

   Possible hint: You will likely find it helpful to use the subspace topology on U.

2. Do Exercise 4.12. You may assume you are working over the field $\mathbb{C}$.

   The following information will likely be useful:

   An affine conic is a degree 2 plane curve of the following form:

   $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

   Note that this general form does not force the curve to be irreducible, which you will likely need to take into account as you do this problem.

   An affine transformation on $\mathbb{A}^2$ can be written in terms of an inverible matrix M and a translation:

   $$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto M \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

3. Compute the multiplicities at the points of intersection of the projectivizations of the plane curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Does the outcome match the prediction of Bézout's Theorem?

4. This question works through Exercise 6.29, which is a warm-up for Grassmannians. It uses the idea that the affine cone of linear subvariety of $\mathbb{P}^3$ corresponds to a subspace of a 4-dimensional vector space.

(a) Consider the line $V(x+y+z+w, x-y+z-w)$ in $\mathbb{P}^3$. It corresponds to the following 2×4 matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Explain why doing row operations to this matrix and converting the rows back into linear equations gives us back the same line in $\mathbb{P}^3$.

(b) Consider the point $p := [1 : 0 : 0 : 0]$ in $\mathbb{P}^3$. Does the line from the previous part contain p? Generally, describe conditions for a line in $\mathbb{P}^3$ to contain or not contain p in terms of the corresponding 2×4 matrices.

(c) Find two distinct lines in $\mathbb{P}^3$ that don't contain p and are also skew, i.e. have empty intersection in $\mathbb{P}^3$.

(d) There is a unique line that contains p and intersects each of the lines you found in the previous part. Find it. Why is it unique?

(e) In general, given any two skew lines in $\mathbb{P}^3$ not containing p, how can you find the unique line that contains p and intersects each of the skew lines?