Math 818, Fall 2022, Dr. Honigs
Homework 4
Due Wed. Oct. 12 at the start of class

**Instructions:** You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

Reminder: Do not use terms like “clear”, “obvious”, or “easy” in your write-up.

---

Definition: Let $R$ be a ring and $S$ a multiplicatively closed subset of $R$. The localization $R[S^{-1}]$ is a ring. It is the set of fractions

$$\left\{ \frac{a}{b} \;\middle|\; a \in R, b \in S \right\}$$

under the equivalence relation defined by

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{iff} \quad \text{there exists } s \in S \text{ such that } s(ad - bc) = 0.$$

The addition and multiplication operations are given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

---

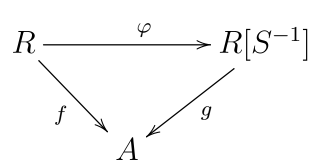
1. Let $R$ be a ring and $\mathfrak{p}$ be a prime ideal in $R$, and $\varphi$ be the following map:

$$\begin{aligned} \varphi : R &\to R_{\mathfrak{p}} \\ r &\mapsto \frac{r}{1} \end{aligned}$$

Show that there is a correspondence between prime ideals of $R_{\mathfrak{p}}$ and prime ideals in $R$ that are contained in $\mathfrak{p}$ (i.e. a bijection between these two sets).

2. In this question you will show that the construction of localization above has the universal property described in class.

Let $S$ be a multiplicatively closed subset of a ring $R$. Let $\varphi : R \to R[S^{-1}]$ be the ring homomorphism where $\varphi(r) = \frac{r}{1}$. Let $f : R \to A$ be a ring homomorphism that maps every element in $S$ to a unit in $A$. Show that there is a unique ring homomorphism $R[S^{-1}] \to A$ that makes the following diagram commute:




Start by deciding how $g$ must act. Since the underlying set of $R[S^{-1}]$ is defined in terms of an equivalence relation, it will be important to show that $g$ is well-defined.

3. (a) Consider the multiplicative set $S := \{1, x, x^2, \ldots\}$ in $\mathbb{C}[x]$. Show that $C[x][S^{-1}]$ is not isomorphic to $\mathbb{C}(x)$. (Hint: Find an element that isn't a unit in $C[x][S^{-1}]$.)

   (b) Use the answer to the previous question (in combination with other results) to show there is a map $g : \mathbb{C}(x)[y] \to \mathbb{C}[x, y]_{(y)}$ so that $g(x) = \frac{x}{1}$ and $g(y) = \frac{y}{1}$. (Note that $(y)$ is a prime ideal so this notation means we're localizing at the set $S := R \setminus (y)$.) Then, show that $g$ is not an isomorphism. Possible hint: Find an element that's a unit in $\mathbb{C}[x, y]_{(y)}$ but its preimage is not a unit.

   (c) Let $R$ be a ring, $a \in R$ and $S = \{1, a, a^2, \ldots\}$. Show that $R[S^{-1}] \cong R[x]/(ax - 1)$.

4. Let $R$ be a ring, $I$ be an ideal in $R$ and $\mathfrak{p}$ be a prime ideal that contains $I$. Show that:

$$(R/I)_{\mathfrak{p}} \simeq R_{\mathfrak{p}}/I_{\mathfrak{p}}.$$

   A caveat on the notation: On the left-hand side $\mathfrak{p}$ should be thought of as the corresponding ideal in $R/I$.

5. Suppose we have a ring homomorphism $\varphi : B \to A$ Let $\mathfrak{p}$ a prime ideal in $A$ and $\varphi^{-1}(\mathfrak{p}) =: \mathfrak{q}$.

   (a) Use the universal property of localization to show that there is a map $B_{\mathfrak{q}} \to A_{\mathfrak{p}}$.

   (b) Prove the map $B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ from the previous part can be used to induce a map $B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.