Math 818, Fall 2022, Dr. Honigs
Homework 5
Due Fri. Oct. 21 at the start of class

Instructions: You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

Reminder: Do not use terms like "clear", "obvious", or "easy" in your write-up.

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1. *Localizing at zero-divisors.* Find the elements in the localization of the ring $\mathbb{Z}/6\mathbb{Z}$ at the multiplicative set $\{1, 2, 4\}$. How many are there?

2. *Projective space and intersection multiplicity.*

   Consider the following curves in $\mathbb{A}^2_{\mathbb{C}}$: $y - x^2 = 0, \quad y - x^2 + 2x - 1 = 0$.

   Homogenize the curves so that they are curves in $\mathbb{P}^2_{\mathbb{C}}$ and then verify that Bézout's Theorem holds for the intersection of these two curves. That is, find all points of intersection and intersection multiplicities.

3. *Practice with affine morphisms.*

   (a) Consider the morphism $\psi : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ that sends a to a^2. What is the pullback ψ^*?

   (b) Consider the morphism $\phi : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$ that sends (a_1, a_2) to (a_1, a_2, a_2^2). What is the pullback ϕ^*?

   (c) Let $W \subset \mathbb{A}^3_{\mathbb{C}}$ be the vanishing of the equations $x_2 - x_1^2, x_3 - x_2^2$ and $V \subset \mathbb{A}^2_{\mathbb{C}}$ be the vanishing of $y_2 - y_1^2$. Show the map ϕ defined in the last part gives a map of varieties $\phi : V \to W$ by showing $\phi(V) \subseteq W$. What is the pullback ϕ^*?

4. *Residue field map.* Consider the morphism $\psi : \mathbb{A}^1_{\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}}$ that sends a to a^2 as a map $\operatorname{Spec} \mathbb{Q}[y] \to \operatorname{Spec} \mathbb{Q}[x]$. In $\mathbb{Q}[y]$, the maximal ideal corresponding to $\pm\sqrt{3}$ is $y^2 - 3$. What is its preimage under ψ^*? Does that make sense given the definition of ψ? What is the map of residue fields that ψ^* induces at these points?

5. *Rational maps* A rational map of affine varieties $f : X \dashrightarrow Y$ over k is dominant if and only if the map $k[Y] \to k(X)$ is injective.

   Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be dominant rational maps of affine varieties over k. Show that the composition $g \circ f$ is also dominant.

6. *Birationality example* Let $g \in k[x_1, \ldots, x_n]$ be a nonzero polynomial and let $V := \operatorname{Spec}(k[x_1, \ldots, x_n]_g)$. Show that $\mathbb{A}^n_k$ is birational to the variety V. Possible hint: You might find it helpful to write the coordinate ring of V as a quotient ring using a result from the previous problem set.

7. *Homogeneous ideals.* Let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. Show that I is a prime ideal if and only if for any forms $F, G \in k[x_0, \ldots, x_n]$, if $FG \in I$ then $F \in I$ or $G \in I$.