Math 818, Fall 2022, Dr. Honigs
Homework 6
Due Wed. Nov. 2 at the start of class

Instructions: You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

Reminder: Do not use terms like "clear", "obvious", or "easy" in your write-up.

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1. Let A, B be rings and $\phi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ be the map given taking preimages under the ring homomorphism $\phi^* : B \to A$. Prove that preimage of a closed set in $\mathrm{Spec}(B)$ is closed in $\mathrm{Spec}(A)$. (We worked through this in class, so this problem is for reviewing the argument.)

2. In this question, you will show how a rational map on affine varieties can be "resolved" into a regular map of affine varieties. Since a rational map is only defined on an open subset of its domain, we can turn it into a regular map by restricting the domain.

   Let V be the affine variety $V = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\} \subseteq \mathbb{A}^2_{\mathbb{C}}$. Consider the rational map $\rho : V \dashrightarrow \mathbb{A}^2$ where
   $$\rho(a_1, a_2) = \left(\frac{a_1}{1 + a_2^2}, \frac{a_1 a_2}{1 + a_2^2} \right).$$

   (a) Let $\mathbb{C}[y_1, y_2]$ be the coordinate ring of $\mathbb{A}^2_{\mathbb{C}}$. What is the pullback map $\rho^* : \mathbb{C}[y_1, y_2] \to \mathbb{C}(V)$?

   (b) Find one polynomial $g \in \mathbb{C}[V]$ where ρ is defined on $D(g) \subseteq V$. (Hint: product of denominators.)

   (c) Use problem 3(c) from homework 4 to show that $D(g)$ is an affine variety in $\mathbb{A}^3$. Call it W.

   (d) Composing the natural map $W \to V$ with ρ gives a regular map $W \to \mathbb{A}^2$. Write what this new map $W \to \mathbb{A}^2$ does to points on algebraic sets. Also show what the pullback map is doing.

3. The Veronese embedding is described in Example 1.28 of I.4.4 in Shafarevich.

   In this question we will prove what the image of the second Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$ is. The point here is to follow Shafarevich's description in a specific example.

   (a) Pick coordinates on your domain and codomain, and write the relations (1.37) in this case. Let's call the vanishing of these relations V.

   (b) Show that for any point in V, at least one of the coordinates corresponding to a square must be nonzero.

(c) Finally, give the map $V \to \mathbb{P}^2$. You may need to give the map on open sets of the domain and then show they agree where they overlap. Show this is an inverse to the Veronese embedding $\mathbb{P}^2 \to V$ – it's enough for the purposes of this question to just check this on points, I won't make you mess around with the coordinate rings here.

4. The image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is $xy - zw = 0$.

   Find the symmetric matrix associated with this quadric and diagonalize it. Find its signature (over $\mathbb{R}$) and its rank.

5. Consider the branched cover $\phi : X = \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} = Y$ that sends a to $a^2 + 2a$.

   (a) Show that $k[X]$ is a finitely generated $k[Y]$-module via ϕ^*. Produce a generating set and show why it works.

   (b) Where is the cover branched? That is, where does it have just one preimage rather than two? (This question is meant to be checked with elementary methods. Also Vakil p. 211 has a nice branched cover graphic.)