Math 818, Fall 2022, Dr. Honigs
Homework 7
Due Wed. Nov. 16 at the start of class

Instructions: You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

Reminder: Do not use terms like "clear", "obvious", or "easy" in your write-up.

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1. Let F/k be a field extension. Show that $z_1, \ldots, z_d \in F$ are algebraically independent if and only if each extension $k(z_1, \ldots, z_i)/k(z_1, \ldots, z_{i-1})$ is transcendental.

2. Let, X, Y be affine varieties over k so that $Y \subseteq X$ (that is, Y is a subvariety of X). Show that if Y has Krull dimension 1 less than the Krull dimension of X, then Y must be a hypersurface in X.

3. Let $V \subseteq \mathbb{A}^5$ be the affine variety given by the vanishing of the irreducible polynomial $x_1^2 + x_2^2 + x_3^2 - 1$. That is, we should think of this polynomial as inside $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]$. Find a transcendence basis $x_{i_1}, \ldots, x_{i_d}$ for $\mathbb{C}(V)$ and express $\mathbb{C}(V)$ explicitly as an algebraic extension over $\mathbb{C}(x_{i_1}, \ldots, x_{i_d})$. That is, for each variable not in the transcendence basis, give its minimal polynomial as an extension of $\mathbb{C}(x_{i_1}, \ldots, x_{i_d})$.

4. Prove that Zariski's lemma is implied by Noether normalization.

   Zariski's Lemma: Let L/k be a field extension. If L is finitely generated as a k-algebra, then it is finitely generated as a k-module.

   Noether normalization: Let X/k be an affine variety so that $\dim(X) = n$. Then there is a finite map $X \to \mathbb{A}_k^n$.

   Note: You will very much need to use the fact that if $A \to B$ is an integral extension and B is a field, then A must be a field as well. Prove this. Hint: We did it in class. The idea is to show that if $a \in A$, then $\frac{1}{a}$ must be in A as well.

5. A *line* in $\mathbb{P}^2_{\mathbb{C}}$ is any projective variety given by the vanishing of a degree one homogeneous polynomial in $\mathbb{C}[x_0, x_1, x_2]$. Show that any two distinct lines in $\mathbb{P}^2$ intersect at exactly one point. (Possible hint: This can be solved using rank-nullity for matrices. You may find it easier to consider solutions in $\mathbb{A}^3$ and then interpret what that says about projective space.)

6. Consider the following conic in $\mathbb{P}^2$:

$$Ax^2 + By^2 + 2Cxy + 2Dxy + 2Eyz + Fz^2 = 0.$$

   (a) So far we've only thought about singularity on affine charts. What does it mean for the conic to have singularities or non-singularities on each standard affine chart?

(b) Show that the conic has no singular points if and only if the corresponding symmetric matrix Q has rank 3.

(c) Show that the set of singular points on the conic consists of those points with coordinates in $\ker(Q)$.

(d) Finally, suppose our conic is smooth and let p be a point on it, which we may write in vector form. Show that the tangent space of the conic at p is equal to $\ker(p^T Q)$.

This is now the end of the problem set. :)