Math 818, Fall 2022, Dr. Honigs
Homework 8
Due Dec. 2

Instructions: You are encouraged to work in groups, but your final written solutions must be in your own words. At the top of your paper, write down the names of anyone you have worked with on the problem set.

Reminder: Do not use terms like “clear”, “obvious”, or “easy” in your write-up.

---

1. Let A be a discrete valuation ring. Show that A has a unique maximal ideal. (Hint: Show that everything in the complement of your maximal ideal invertible.)

2. We showed in class that if $(R, \mathfrak{m})$ is a Noetherian, local integral domain that is regular of dimension 1, then $\mathfrak{m} = (t)$ for some $t \in \mathfrak{m}$. It was stated that we may define a valuation on R by mapping any $r \in \mathrm{Frac}(R)$ to the unique $n \in \mathbb{Z}$ so that $r = ut^n$ for some unit u in R. Show that this in fact defines a discrete valuation.

3. Suppose we have an elliptic curve $y^2 = (x - e_1)(x - e_2)(x - e_3)$ over $\mathbb{C}$ where $e_1, e_2, e_3 \in \mathbb{C}$ are distinct.
   (a) Consider this curve to be in $\mathbb{P}^2$. What is the homogeneous equation for it? What are the (projective) coordinates for the unique point at infinity?
   (b) Let $P_i := [e_i : 0 : 1]$ and P_∞ by the unique point at infinity. Show that the principal divisor $\mathrm{div}(y)$ is equal to $[P_1] + [P_2] + [P_3] - 3[P_\infty]$. (We will solve a similar problem Friday Nov. 18 in lecture.)

4. *Practice with projective space.* In this question we will consider the topology on the *points* in $\mathbb{P}^n_{\mathbb{C}}$. Note we will generally be considering the Zariski topology on points in this question, not on prime ideals, and we're just working over $\mathbb{C}$ for convenience and familiarity. Recall that closed sets in $\mathbb{P}^n$ may all be written as $V(I)$ where I is some *homogeneous* ideal in $\mathbb{C}[x_0, \ldots, x_n]$.

   Let $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a projective variety and let $U_0 \simeq \mathbb{A}^n_{\mathbb{C}}$ be the standard affine chart where $x_0 \neq 0$.
   (a) The intersection $V \cap U_0$ is a closed set in $\mathbb{A}^n_{\mathbb{C}}$, so it must be the vanishing of an ideal $J \subseteq \mathbb{C}[x_1, \ldots, x_n]$. The homogenization of J, $J^h \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is the ideal generated by the homogenizations of all the elements in J. The <u>*projective closure*</u> of $V \cap U_0$, i.e., the smallest closed set in $\mathbb{P}^n$ containing $V \cap U_0$, is $\overline{V \cap U_0} := V(J^h)$. Prove (directly, using ideals) that $V \cap U_0 = V(J^h) \cap U_0$.
   (b) Show that either $V \cap U_0 = \emptyset$ or $V = \overline{V \cap U_0}$. (Hint: Varieties are irreducible.)
   (c) Show that closed (affine) subvarieties of $\mathbb{A}^n$ are in one-to-one correspondence with closed subvarieties of $\mathbb{P}^n$ that have nonempty intersection with U_0.

(d) Describe all of the closed subvarieties of $\mathbb{P}^1_{\mathbb{C}}$. (Hint: you may use your knowledge of all of the closed subvarieties of $\mathbb{A}^1_{\mathbb{C}}$.)

(e) Exam practice, not to be graded: In class we proved that the closed subvarieties of $\mathbb{A}^2_{\mathbb{C}}$ consist of all of $\mathbb{A}^2_{\mathbb{C}}$, $\emptyset$, points, and the vanishings of principal ideals generated by irreducible varieties. Is it possible to make a similar statement about $\mathbb{P}^2_{\mathbb{C}}$?

5. In this question we will consider some points and lines in $\mathbb{P}^3_{\mathbb{C}}$. This problem is a great warm-up for the lectures on Grassmannians. It heavily uses the idea that linear subvarieties of $\mathbb{P}^3$ projective space correspond to subspaces of a 4-dimensional vector space V.

   (a) Consider the line $V(x+y+z+w, x-y+z-w)$ in $\mathbb{P}^3$. It corresponds to the following 2×4 matrix:

   $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

   Explain why doing row operations to this matrix and converting the rows back into linear equations gives us back the same line in $\mathbb{P}^3$.

   (b) Consider the point $p := [1:0:0:0]$ in $\mathbb{P}^3$. Does the line from the previous part contain p? Generally, describe conditions for a line in $\mathbb{P}^3$ to contain or not contain p in terms of the corresponding 2×4 matrices.

   (c) Find two distinct lines in $\mathbb{P}^3$ that don't contain p and are also skew, i.e. have empty intersection in $\mathbb{P}^3$.

   (d) There is a unique line that contains p and intersects each of the lines you found in the previous part. Find it. Why is it unique?

   (e) In general, given any two skew lines in $\mathbb{P}^3$ not containing p, how can you find the unique line that contains p and intersects each of the skew lines?

6. Recall that may consider the space $\mathbb{P}^5_{\mathbb{C}}$ to parametrize conics in $\mathbb{P}^2_{\mathbb{C}}$: For our convenience, we may decide a point $[a_0 : \cdots : a_5]$ corresponds to a conic (we could also do this without the 2's, they're just there for convenience):

   $$a_0x^2 + a_1y^2 + 2a_2xy + 2a_3xz + 2a_4yz + a_5z^2.$$

   (a) Show that the singular (i.e. non-smooth) conics form a hypersurface in $\mathbb{P}^5$. What is the equation for the hypersurface? (Hint: Homework 7, Exercise 6.)

   (b) Show that all the conics passing through a given point in $\mathbb{P}^2$ form a hyperplane in $\mathbb{P}^5$.

   (c) Consider the points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. Give equations for the conics in $\mathbb{P}^5$ that pass through each of those points.

   (d) Find another point in $\mathbb{P}^2$ that is not collinear with any two of the points from the last part.

(e) Give equations for the conics in $\mathbb{P}^5$ that pass through all four of these points. It should be possible to parametrize these by a $\mathbb{P}^1$. Does this seem reasonable?

(f) Use your answer to part (a) to find all of the singular conics in your $\mathbb{P}^1$'s worth of conics passing through four points. What do they look like?

(g) Not to be graded, but for your consideration: Why has our work shown it is true in general that the family of conics in $\mathbb{P}^2$ passing through any four points where no three are contained in the same line are parametrized by a $\mathbb{P}^1$ and must contain at least one singular conic? (Hint: start by considering projective transformations). Another fun thing to consider: what we've done here is very related to the slogan "A conic is uniquely determined by 5 points". Is this slogan totally precise?