

Derived Equivalent Varieties and their Zeta Functions

by

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Abstract

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In order to shed light on Orlov’s conjecture that derived equivalent smooth, projective varieties have isomorphic Chow motives, we examine the zeta functions of derived equivalent varieties over finite fields; in this setting Orlov’s conjecture predicts equality of zeta functions. It is demonstrated that derived equivalent smooth, projective varieties over finite fields that are abelian or satisfy a certain condition on their cohomology. This condition is satisfied, for example, by a surface or Calabi–Yau 3–fold.

One of our approaches to comparing the zeta functions of derived equivalent varieties over finite fields comes from using the Lefschetz Fixed Point Theorem to turn the question into one of comparing the ℓ -adic étale cohomology of varieties. Cohomology groups are not in general preserved under the action of Fourier–Mukai equivalences on cohomology, but cohomological structures we call even and odd Mukai–Hodge structures, which are a realization of the Mukai motive, are preserved. Investigation into when isomorphism of these cohomological structures implies equality of zeta functions gives us our cohomological condition for equality of zeta functions.

We also develop a relative version of the map Fourier–Mukai transforms induce on cohomology and define a relative notion of even and odd Mukai–Hodge structures, and show these structures are preserved in a situation arising from the derived equivalence of smooth, projective varieties with semiample (anti-)canonical bundles. Using this result, it is demonstrated that when derived equivalent smooth, projective varieties have semiample (anti-)canonical bundles, the fibers over any fixed geometric point in their shared (anti-)canonical variety must also have isomorphic even and odd Mukai–Hodge structures. Hence, for any such varieties over finite fields, if their geometric fibers satisfy any of the conditions identified for isomorphism of Mukai–Hodge structures to imply equality of zeta functions, then the varieties themselves also have equal zeta functions.

For my grandmother, Margaret Honigs.

Contents

Contents	ii
0 Introduction	1
0.1 Notation	7
1 Motives	8
1.1 Pure motives	8
1.2 Weil cohomology theories	14
1.3 Lefschetz fixed-point and zeta functions	20
1.4 The Mukai motive	25
2 Fourier–Mukai transforms and their properties	28
2.1 Fourier–Mukai transforms: definition	28
2.2 Existence and uniqueness of Fourier–Mukai transforms and natural transformations between them	29
2.3 Fourier–Mukai transforms and canonical bundles	31
2.4 Fourier–Mukai transforms and fibers	36
3 Isomorphism of Mukai–Hodge structures, and its consequences	41
3.1 Fourier–Mukai transforms acting on cohomology	41
3.2 Isomorphism of Mukai–Hodge structures...	43
3.3 ...and its consequences	45
4 Abelian Varieties	51
4.1 Derived equivalent abelian varieties have equal zeta functions	51
4.2 Mukai–Hodge structures and abelian varieties	53
5 Isomorphism of relative Mukai–Hodge structures, and its consequences	56
5.1 Relative Mukai–Hodge structures	56
5.2 Relative Mukai–Hodge structures and zeta functions	60
Bibliography	62

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Chapter 0

Introduction

The formalism of derived categories, and in particular the notion of the derived category of the abelian category of coherent sheaves on a variety, was introduced in the 1960's by Verdier and his advisor Grothendieck [47] as part of Grothendieck's theory of duality, and with the intention that these categories should be the correct setting for homological algebra. Recently, derived categories have emerged as important objects in their own right, which package cohomological information about the varieties in question in a surprisingly manageable way. In particular, the bounded derived category of the abelian category of coherent sheaves on a variety X , abbreviated from here on to the “derived category of X ” and denoted $D^b(X)$, is an invariant of varieties that reveals a great deal of geometric information about the variety X .

Varieties X and Y are *derived equivalent* if there is an exact equivalence of triangulated categories between $D^b(X)$ and $D^b(Y)$. Derived equivalence between smooth, projective varieties preserves properties such as dimension, Kodaira dimension and order of the canonical bundle, and in fact the canonical rings themselves (see Chapter 2 for more in this vein). If A is an abelian variety, the isomorphism class of the product of A with its dual (but not of A itself) is determined by $D^b(A)$. Connections between derived equivalence and moduli theory have appeared as well. For instance, any variety derived equivalent to a K3 surface X can be shown to be not only another K3 surface, but a connected component of a moduli space classifying semi-stable sheaves on X ; the characteristic 0 case was completed in work by Mukai [31] and Orlov [33], and the characteristic p case was completed more recently by Lieblich and Olsson [26] and Ward [50]. The question of how much information the derived category carries is still open; quite recently, it was shown, in both cases using derived equivalent Calabi–Yau 3-folds as examples, that derived equivalence does not necessarily preserve varieties' fundamental groups (Bak [5], Schnell [40]) or Brauer groups (Addington [2]).

Derived categories are closely related to other areas of algebro-geometric research as well, such as birational geometry and the minimal model program, mirror symmetry, and motivic questions, such as the Tate and Hodge conjectures. Furthermore, Orlov has conjectured that for any smooth, projective variety X , its derived category determines its Chow motive with rational coefficients:

Conjecture 0.0.1 ([36, Conjecture 1]). Let X and Y be smooth projective varieties such that $D^b(X)$ and $D^b(Y)$ are equivalent as triangulated categories. Then the Chow motives of X and Y with rational coefficients are isomorphic in the category of effective Chow motives, or, equivalently if X and Y are defined over a field admitting resolution of singularities, in Voevodsky's category of geometric motives [49].

Since much of the work in this thesis is over finite fields, we will use Chow motives with rational coefficients (see Definition 1.1.16), rather than geometric motives. There is a fully faithful functor from the category $C_{\sim \text{rat}} V(k)_{\mathbb{Q}}^0$ (see Definition 1.1.9) into the category of Chow motives with rational coefficients. The category $C_{\sim \text{rat}} V(k)_{\mathbb{Q}}^0$ has objects given by smooth, projective varieties over a field k , and the Chow motive of a smooth, projective variety X is the image of X via the functor from $C_{\sim \text{rat}} V(k)_{\mathbb{Q}}^0$ to the category of Chow motives. Given two smooth projective varieties X and Y , the morphisms from X to Y in $C_{\sim \text{rat}} V(k)_{\mathbb{Q}}^0$ are given by elements of $\mathcal{Z}_{\sim \text{rat}}^{\dim X}(X \times Y)_{\mathbb{Q}}$, cycles in the Chow group of $X \times Y$ that are of codimension $\dim X$ (see Section 1.1 for the definition of cycles and Chow groups). Given smooth, projective varieties $X, Y, Z \in \text{ob } C_{\sim \text{rat}} V(k)_{\mathbb{Q}}^0$ and morphisms from X to Y and Y to Z , $\alpha \in \mathcal{Z}_{\sim \text{rat}}^{\dim X}(X \times Y)_{\mathbb{Q}}$ and $\beta \in \mathcal{Z}_{\sim \text{rat}}^{\dim Y}(Y \times Z)_{\mathbb{Q}}$, their composition is given by $\pi_{13*}(\pi_{12}^* \alpha \cdot \pi_{23}^* \beta)$, where π_{ij} is the projection from $X \times Y \times Z$ to the product of its i^{th} and j^{th} factors and \cdot denotes the intersection product. So, an isomorphism between the Chow motives of smooth, projective varieties X and Y is given by a cycle class in the Chow group of $X \times Y$ of codimension $\dim X$ that is an isomorphism in $C_{\sim \text{rat}} V(k)_{\mathbb{Q}}^0$.

Conjecture 0.0.1 has been proved in several cases. For example, in the case where the derived equivalent varieties have ample or anti-ample canonical bundles, we have the following theorem:

Theorem 2.3.5 (Bondal and Orlov [8, Theorem 2.5]). *If there is an exact equivalence $D^b(X) \simeq D^b(Y)$ between smooth varieties X and Y , and X is projective and has ample or anti-ample canonical bundle, then X is isomorphic to Y .*

The cycle class of a graph of an isomorphism of varieties gives an isomorphism of motives; the cycle class of the graph of the inverse of the isomorphism of varieties gives the inverse in the category of motives. Hence, it is an immediate consequence of the Theorem 2.3.5, derived equivalent varieties with ample or anti-ample canonical bundles have isomorphic motives.

Many of the exact functors $F : D^b(X) \rightarrow D^b(Y)$ between the derived categories of smooth, projective varieties X and Y , including all equivalences (due to Orlov's [35, Theorem 3.2.1], quoted here in Theorem 2.2.3) are isomorphic to a functor of the form controlled by an object $P \in D^b(X \times Y)$, called its kernel, in the sense that there is an isomorphism between F and $\mathbb{R}p_{Y*}(\mathbb{L}p_1^*(-) \otimes^{\mathbb{L}} P)$, where p_X and p_Y are the projections $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$, and pullback, pushforward and tensor are in their derived versions. Such functors are called Fourier–Mukai transforms. We will call Fourier–Mukai transforms that give equivalences *Fourier–Mukai equivalences*.

Conjecture 0.0.1 has also been verified in the case where the derived equivalence $D^b(X) \rightarrow D^b(Y)$ between smooth, projective varieties is isomorphic to a Fourier–Mukai transform with a kernel meeting a certain criterion:

Theorem 0.0.2 (Orlov [36, Theorem 1]). *Let $F : D^b(X) \rightarrow D^b(Y)$ be a Fourier–Mukai transform between the derived categories of smooth, projective varieties where $\dim(X) = \dim(Y) = n$. Suppose the kernel P of the Fourier–Mukai transform has support of dimension n . If F is fully faithful then the Chow motive of X is a direct summand of that of Y , and if F is an equivalence of categories, then the motive of X is isomorphic to the motive of Y .*

Given a Fourier–Mukai transform $D^b(X) \rightarrow D^b(Y)$ with kernel $P \in D^b(X \times Y)$, the Mukai vector of P , denoted $v(P)$ is the element of the Chow group of $X \times Y$ given by $\text{ch}(P) \cdot \sqrt{\text{td}(X \times Y)}$ (see Definition 3.1.1 as well as Chapter 1 for the definition of the Chern character and Todd class). Theorem 0.0.2 is proved by showing that, under its hypotheses, the cycle class in the Chow group of $X \times Y$ of the Mukai vector of a Fourier–Mukai equivalence is an isomorphism between the Chow motives (with rational coefficients) of X and Y .

However, without the restriction on the dimension of the support given in Theorem 0.0.2, the Mukai vectors of a kernel of a Fourier–Mukai equivalence is not in general a cycle of codimension $\dim(X)$ on $X \times Y$, and so its cycle class in the Chow group of $X \times Y$ does not give well-defined maps between the Chow motives of X and Y .

The cycle classes of Mukai vectors of Fourier–Mukai transforms do always give maps between *Mukai motives* (Definition 1.4), and any derived equivalent smooth and projective X and Y have isomorphic Mukai motives [36, Proposition 1]. These Mukai motives are defined in the category of Chow motives, but are not in the image of $C_{\sim_{\text{rat}}} V(k)_{\mathbb{Q}}^0$ or in the full subcategory of effective Chow motives in the category of Chow motives (see Definition 1.1.13), since Tate twists must be introduced in order to define them (see Definition 1.1.16 and Section 1.2 for a more information on Tate twists).

This thesis takes a more numerical approach toward collecting evidence for Conjecture 0.0.1. The zeta function of a variety over a finite field depends only on its Chow motive (see Section 1.3), and so Orlov’s conjecture, if true, would imply that derived equivalent smooth, projective varieties over a finite field have identical zeta functions (we organize some of the ideas at play here in the diagram below). In this dissertation, we prove, strongly using the isomorphism of Mukai motives implied by derived equivalence, that this is the case for some situations: for abelian varieties, for varieties that satisfy a condition on their cohomology that is met by, e.g., any surface or Calabi–Yau 3-fold, and some classes of varieties depending on the semi-ampleness of their canonical or anti-canonical classes and the behavior of their fibers over their (anti-)canonical variety.

If two smooth, projective varieties X and Y over a finite field have equal zeta functions, then the Tate conjectures imply that we can produce a cycle class up to numerical equivalence (see Example 1.1.5) that gives an isomorphism between the numerical motives of X and Y . However, numerical equivalence is a coarser relation on cycles than rational equivalence,

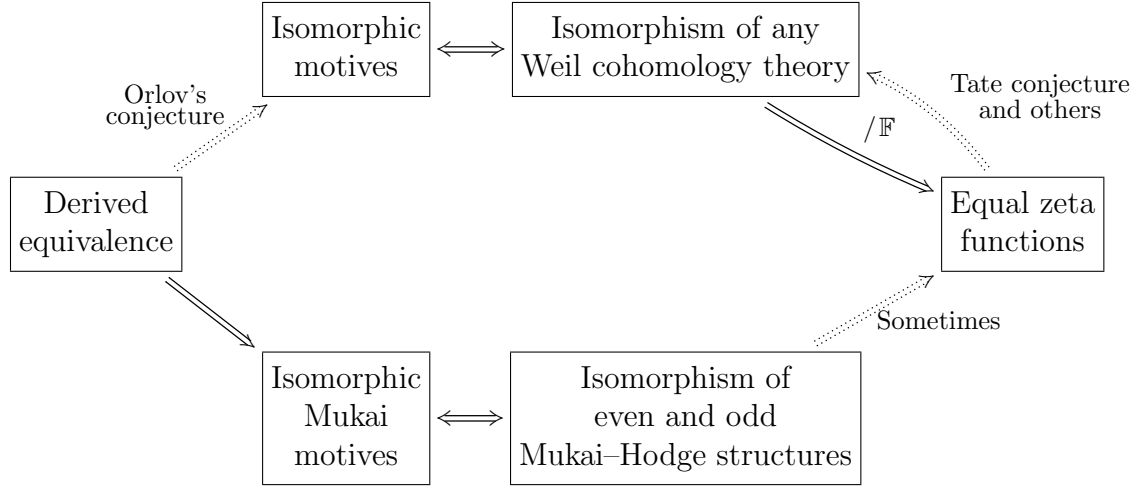


Figure 0.1:

the equivalence relation defining Chow groups. In Section 1.3, we give a discussion of this application of the Tate conjectures and suggest what further conjectures would suffice to produce an isomorphism of Chow motives. Although Section 1.3 suggests a conjectural method for producing an isomorphism of the Chow motives of derived equivalent smooth, projective varieties over finite fields that have equal zeta functions, this method has no dependence on the derived equivalence, and so we are not also conjecturing a functor from the category of derived categories of smooth, projective varieties with exact functors to the category of Chow motives with rational coefficients that maps the derived category of a variety to its Chow motive.

Thesis results

We will prove the following theorems:

Theorem 4.1.5. *Let A and B be abelian varieties defined over a finite field \mathbb{F} . If A and B are derived equivalent, then A and B have equal zeta functions.*

Theorem 3.3.6. *Let X and Y be varieties of dimension d over a finite field \mathbb{F} such that $D^b(X)$ is equivalent to $D^b(Y)$. Let φ be the geometric Frobenius endomorphism. If*

$$\mathrm{Tr}(\varphi^*|H^i(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^*|H^i(Y, \mathbb{Q}_\ell))$$

for $\lfloor \frac{d}{2} \rfloor - 1$ even values and $\lceil \frac{d}{2} \rceil - 1$ odd values of $1 \leq i \leq d$, then $\zeta(X) = \zeta(Y)$.

Smooth, projective surfaces and 3-dimensional varieties with vanishing first cohomology groups (e.g. Calabi–Yau 3-folds) fulfill the hypotheses of Theorem 3.3.6.

Theorem 5.2.1. *Let X and Y be smooth, projective varieties of dimension d over a finite field \mathbb{F}_q such that $D^b(X)$ is equivalent to $D^b(Y)$. Suppose that X , and hence also Y (see Corollary 2.3.13(c)), has a semiample canonical (or anti-canonical) bundle. Let S be the (anti-)canonical variety of X and of Y (see Proposition 2.3.3). If, for each geometric point $s \in S$, the fibers X_s and Y_s fulfill at least one of the following hypotheses, then $\zeta(X) = \zeta(Y)$:*

- (i) X_s and Y_s are smooth, projective varieties such that

$$\mathrm{Tr}(\varphi^*|H^i(X_s, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^*|H^i(Y_s, \mathbb{Q}_\ell))$$

for $\lfloor \frac{d}{2} \rfloor - 1$ even values and $\lceil \frac{d}{2} \rceil - 1$ odd values of $1 \leq i \leq d$, where φ is the geometric Frobenius endomorphism and $d = \dim X_s = \dim Y_s$.

- (ii) X_s and Y_s are abelian varieties of dimension 3 or lower.

Theorem 4.1.5 is shown using [34, Theorem 2.19] of Orlov, which states that if A and B are derived equivalent varieties, then $A \times \hat{A} \cong B \times \hat{B}$, as well as the fact that abelian varieties over finite fields are isogenous if and only if they have equal zeta functions (Tate [43, Theorem 1, Section 3]).

To prove Theorem 3.3.6, we use a more general approach, which we outline here:

Outline of proof of Theorem 3.3.6. By the Lefschetz fixed-point theorem and the Riemann hypothesis portion of the Weil conjectures, we have that $\zeta(X) = \zeta(Y)$ if and only if

$$\mathrm{Tr}(\varphi^*|H^i(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^*|H^i(Y, \mathbb{Q}_\ell))$$

for all i , where φ is the geometric Frobenius morphism and H is ℓ -adic étale cohomology.

When the varieties in question are smooth and projective, derived equivalences can be realized as Fourier–Mukai functors, which induce maps on Weil cohomology theories (ℓ -adic étale cohomology is an example of one of these). These maps are not necessarily degree-preserving, and, when acting on Weil cohomology theories, they introduce Tate twists, which we must keep track of in order that the map on cohomology be compatible with the action of geometric Frobenius.

However, two cohomological structures are preserved under this map: the even and odd Mukai–Hodge structures, which are cohomological realizations of Mukai motives. These structures, for a variety X , are, respectively:

$$\bigoplus_{i=0}^{d_X} H^{2i}(X/K)(i), \quad \bigoplus_{i=1}^{d_X} H^{2i-1}(X/K)(i)$$

Along with the symmetry among cohomology groups from Poincaré duality or the hard Lefschetz theorems, the preservation of these Mukai–Hodge structures under derived equivalences gives us the condition for equality of zeta functions described in Theorem 3.3.6. \square

We were also able to use the methods of the proof of Theorem 3.3.6 to show that zeta functions of derived equivalent 3-dimensional abelian varieties over finite fields must be equal, without using [34, Theorem 2.19]:

Proposition 4.2.1. *Let A and B be 3-dimensional abelian varieties over a finite field \mathbb{F}_q . If A and B have isomorphic even and odd Mukai–Hodge structures, then $\zeta(A) = \zeta(B)$.*

Theorem 5.2.1 is the result of applying Theorems 3.3.6 and Proposition 4.2.1 after making a reduction in the case of derived equivalent varieties with semi-ample canonical or anti-canonical bundles. Derived equivalent smooth, projective varieties X and Y have isomorphic canonical and anti-canonical varieties (Orlov [35, Theorem 3.2.1]), and so if X and Y have semi-ample (that is, globally generated) canonical or anti-canonical varieties, we can induce canonical maps from each of them to their shared canonical or anti-canonical variety, giving us more information about comparing X and Y . When X and Y are defined over a finite field \mathbb{F}_q , we prove that a relative version of the Mukai–Hodge structures is preserved: if $f : X \rightarrow S$ and $g : Y \rightarrow S$ are the canonical maps, we have the isomorphisms

$$\bigoplus_i R^{2i} f_* \mathbb{Q}_\ell(i) \cong \bigoplus_i R^{2i} g_* \mathbb{Q}_\ell(i) \quad \text{and} \quad \bigoplus_i R^{2i-1} f_* \mathbb{Q}_\ell(i) \cong \bigoplus_i R^{2i-1} g_* \mathbb{Q}_\ell(i).$$

Localizing these isomorphisms at any geometric point s in S shows that the fibers X_s and Y_s have isomorphic Mukai–Hodge structures. And so we can formulate Theorem 5.2.1: if all the fibers X_s and Y_s satisfy the hypotheses of Theorem 3.3.6 or Proposition 4.2.1, then these pairs of fibers each have equal zeta functions, and so do X and Y .

Outline

Chapter 1 is focused on introducing the ideas at play in Figure 0. We define several categories of pure motives, focusing on Chow motives and their universal property classifying Weil cohomologies, as well as the implications of the Tate conjectures in relating motives to zeta functions. We give the data and axioms of a Weil cohomology theory as well as a discussion of the background we will need to relate zeta functions to them, including a discussion of the Lefschetz fixed-point theorem and the various types of Frobenius morphisms. Chapter 1 concludes with a discussion of Mukai motives.

Chapter 2 gives background on derived equivalence. It starts out with a discussion of Fourier–Mukai functors and a short literature survey about the existence and uniqueness of Fourier–Mukai kernels in Sections 2.1 and 2.2. Then Section 2.3 gives a discussion about the canonical bundles of derived equivalent varieties, particularly working through different proofs of several results from Toda’s [46]. Given varieties X, Y, S and morphisms $f : X \rightarrow S$, $g : Y \rightarrow S$, Section 2.4 gives a criterion for when Fourier–Mukai equivalence of X and Y implies Fourier–Mukai equivalence of the fibers X_s and Y_s for geometric points $s \in S$.

In the beginning of Chapter 3, we introduce even and odd Mukai–Hodge structures, demonstrate that they are preserved under derived equivalence, and then focus on finding

cases where we can fill in the arrow labeled “sometimes” above, which is the heart of what is being proved in this thesis. That is, we examine what information about comparing the zeta functions of derived equivalent varieties can be extracted from the Mukai–Hodge structures, culminating in the proof of Theorem 3.3.6.

Chapter 4 explores comparing zeta functions of abelian varieties. We give a proof of Theorem 4.1.5 and some exposition on the theories involved in that proof. We also examine the consequences of the preservation of even and odd Mukai–Hodge structures in the abelian varieties case, and demonstrate that the strategy for proving Theorem 3.3.6 can be used, along with some additional combinatorial argument, to show that abelian varieties of dimension 3 have equal zeta functions.

In Chapter 5, we apply the results from Section 2.4 to the problem of comparing zeta functions of derived equivalent varieties over finite fields, proving Theorem 5.2.1. In order to do this, we develop a relative version of the map that Fourier–Mukai functors induce on cohomology, and introduce a notion of even and odd Mukai–Hodge structures, and show these structures are preserved in a situation arising from the derived equivalence of smooth, projective varieties with semiample (anti-)canonical bundles.

0.1 Notation

Throughout this thesis, a variety is defined to be an integral, separated scheme of finite type over a field k .

Chapter 1

Motives

The focus of this thesis is the finite fields case of Orlov’s conjecture that derived equivalent smooth, projective varieties have isomorphic Chow motives [36, Conjecture 1], quoted in Conjecture 0.0.1. In this chapter, we will give some exposition on pure motives, Weil cohomology theories, zeta functions, and the relationships between them.

Our treatment of these topics will not by any means be exhaustive; the theory of motives is a large one, and connected with many deep questions. The ideas for the theories of motives and Weil cohomologies arose in pursuit of the Weil conjectures. In Chapter 3, we will use some results relating to Deligne’s proof of Weil conjectures, but not delve into the full statements of the Weil conjectures or standard conjectures here. Our focus will be on pure motives, which are defined for smooth, projective varieties, and will not discuss mixed motives, the more-conjectural generalization to singular varieties.

Some other sources of information on motives include Kleiman’s articles [25, 24], which focus on, respectively, motives and the connection between Weil cohomologies and the standard conjectures, Scholl’s exposition [41] on pure motives, and André’s comprehensive and recent book [3], which gives an introduction to pure motives, mixed motives and periods. For a detailed exposition on \otimes and Tannakian categories, as well as cohomology theories, motives, and the standard conjectures, see Saavedra Rivano [39].

1.1 Pure motives

Conjecture 0.0.1 refers to effective Chow motives with rational coefficients and Voevodsky’s geometric motives, which can be used interchangeably in many situations. Voevodsky [49] defines his category of geometric motives for varieties over a field k , when k is perfect. By [49, 4.2.6], there is a full embedding of the category of effective Chow motives over k into this category of motives when k admits a resolution of singularities (see Friedlander and Voevodsky [19, Definition 3.4]) so in many situations it does not matter whether Chow motives or Voevodsky’s category is being considered: any field of characteristic zero admits a resolution of singularities [19, Proposition 3.5]. However, the work in this thesis focuses

on finite fields, where the resolution of singularities has not been proved in general, so for us the relevant category is Chow motives, and we work up to a definition of them in this section.

Cycles and adequate equivalence relations

In order to define morphisms in the category of motives, we need to define the graded group of algebraic cycles:

Definition 1.1.1. Given a smooth, projective integral variety X of dimension d , $\mathcal{Z}^r(X)$ is the free abelian group generated by irreducible closed codimension- r subvarieties of X : the elements of $\mathcal{Z}^r(X)$ consist of formal sums of such subvarieties with coefficients in \mathbb{Q} . The group of (algebraic) *cycles* on X is the group $\mathcal{Z}(X) := \bigoplus_r \mathcal{Z}^r(X)$, which is graded under the operation of intersection.

In other places in the literature, $C(X)$ is sometimes used to denote cycle groups.

Definition 1.1.2. Let X be a smooth, projective variety and fix commutative ring R . We define the cycle group of X with R -coefficients to be the tensor product

$$\mathcal{Z}(X)_R := \mathcal{Z}(X) \otimes_{\mathbb{Z}} R.$$

Definition 1.1.3. Given smooth, projective varieties X, Y , elements of $\mathcal{Z}^{\dim X + r}(X \times Y)_R$ are called *algebraic correspondences* of degree r with coefficients in R between X and Y .

Any category of pure motives includes a choice of adequate equivalence relation on cycles with R -coefficients. Note that we tensor the cycle group with the chosen coefficient ring before quotienting by an adequate equivalence relation: these two operations do not necessarily commute.

Definition 1.1.4. An *adequate equivalence relation* \sim on the group of cycles with coefficients over a fixed ring R is an equivalence that has the following properties (cf André [3, Definition 3.1.1.1]) for any smooth projective varieties X, Y :

- (1) The relation \sim is compatible with both the R -linear structure and the graded structure of $\mathcal{Z}(X)_R$.
- (2) For any two cycles $\alpha, \beta \in \mathcal{Z}(X)_R$, there exists a cycle that is both equivalent to α , and properly intersects β (see Fulton [20, Section 2.4 and Chapter 7] for a comprehensive discussion of proper intersections).
- (3) Let p_X and p_Y be the standard projections from $X \times Y$ to its first and second factors. For any $\alpha \in \mathcal{Z}(X)_R$ and all $\gamma \in \mathcal{Z}(X \times Y)_R$ that properly intersect $(p_X)^{-1}(\alpha)$, if $\alpha \sim 0$, then $\gamma_*(\alpha) := p_Y(\gamma \cdot p_Y^{-1}(\alpha)) \sim 0$. (\cdot denotes intersection). Condition (3) implies that the assignment of smooth projective varieties to R -algebras sending X to $\mathcal{Z}_{\sim}(X)_R$ is a contravariant functor ([3, Definition 3.1.1.1]).

We denote the group of cycle classes of X with coefficients in R modulo an adequate equivalence relation \sim by $\mathcal{Z}_\sim(X)_R := \mathcal{Z}(X)_R / \sim$.

Example 1.1.5. Two varieties are *rationally equivalent* if they are members of the same family of cycles parametrized by any rational or unirational variety; see Fulton [20] or Kleiman [24] for further discussion.

Homological equivalence is always given with respect to a Weil cohomology theory (see Section 1.2 for a definition) we have chosen *a priori*: two cycles are homologically equivalent if their images inside this Weil cohomology theory are equal.

Two cycles on a variety X are *numerically equivalent* if their respective intersections with any other cycle on X have equal degrees.

The order of these classical examples of adequate equivalence relations on cycles from finest to coarsest is rational equivalence \sim_{rat} , algebraic equivalence, homological equivalence and numerical equivalence. Rational equivalence is the finest of all possible adequate equivalence relations [3, Lemme 3.2.2.1], and numerical equivalence is the coarsest. If R is a \mathbb{Q} -algebra, then Voevodsky's adequate equivalence smash nilpotence [48] is coarser than algebraic and finer than homological equivalence.

Definition 1.1.6. The Chow group is defined to be $\mathcal{Z}(X)/\sim_{\text{rat}}$, and $\mathcal{Z}(X)_R/\sim_{\text{rat}}$ the Chow group with R -coefficients. Elsewhere in the literature, $\mathcal{Z}(X)/\sim_{\text{rat}}$ or $\mathcal{Z}(X)_{\mathbb{Q}}/\sim_{\text{rat}}$ is sometimes denoted by $A(X)$.

Pure motives

We will define several categories in the process of constructing the category of motives.

Definition 1.1.7. Let $V(k)$ be the category of smooth, projective varieties over a field k and the morphisms between them.

Definition 1.1.8. Fix a coefficient ring R and an adequate equivalence relation \sim . The category of correspondences $C_\sim V(k)_R$ has the same objects as $V(k)$. Given $X, Y, Z \in \text{ob } C_\sim V(k)_R$, morphisms in $C_\sim V(k)_R$ from X to Y given by cycles on $X \times Y$ with coefficients in R , modulo \sim :

$$\text{Hom}_{C_\sim V(k)_R}(X, Y) := \mathcal{Z}_\sim(X \times Y)_R.$$

Given $\alpha \in \mathcal{Z}_\sim(X \times Y)_R$ and $\beta \in \mathcal{Z}_\sim(Y \times Z)_R$, their composition in $C_\sim V(k)_R$ is given by $\pi_{13*}(\pi_{12}^* \alpha \cdot \pi_{23}^* \beta)$, where π_{ij} is the projection from $X \times Y \times Z$ to the product of its i^{th} and j^{th} factors and \cdot denotes the intersection product.

Definition 1.1.9. Fix a ring of coefficients R and adequate relation \sim on R -cycles. There is a contravariant functor

$$V(k)^{\text{op}} \rightarrow C_\sim V(k)_R \tag{1.1}$$

sending varieties to themselves and any morphism $f : X \rightarrow Y$ in $V(k)$ to $f^* \in \mathcal{Z}_{\sim}^{\dim X}(Y \times X)_R$, which we define to be the cycle class of the transpose of the graph of f .

The category $C_{\sim}V(k)_R^0$ is given by restricting the morphisms in $C_{\sim}V(k)_R$ to correspondences of degree 0.

Remark 1.1.10. Note that we may define a functor

$$V(k)^{\text{op}} \rightarrow C_{\sim}V(k)_R^0 \quad (1.2)$$

by restricting the codomain of functor (1.1) since the transpose of the graph of a morphism of varieties $f : X \rightarrow Y$ is a correspondence of degree 0 between Y and X .

Definition 1.1.11. A *preadditive category* is a category C such that for any objects $X, Y, Z \in C$, $\text{Hom}_C(X, Y)$ is an abelian category, and the composition map

$$\text{Hom}_C(X, Y) \times \text{Hom}_C(Y, Z) \rightarrow \text{Hom}_C(X, Z)$$

is bilinear.

A *pseudo-abelian category* is a pre-additive category such that every idempotent has a kernel, or equivalently, every idempotent splits.

Definition 1.1.12. Let C be a pre-additive category. Its *pseudo-abelian envelope*, $\text{kar}(C)$, also called, elsewhere in the literature, the Karoubi envelope, the pseudo-abelian completion, or the idempotent completion, has objects given by pairs (X, e) where X is an object in C and e is an idempotent endomorphism on X , and for any $(X, e), (Y, e') \in \text{ob kar}(C)$, $\text{Hom}_{\text{kar}(C)}((X, e), (Y, e')) := \{f \in \text{Hom}_C(X, Y) \mid f = e' \circ f \circ e\}$. The composition of any two morphisms $f : (X, e) \rightarrow (Y, e')$ and $g : (Y, e') \rightarrow (Z, e'')$ is $g \circ f : (X, e) \rightarrow (Z, e'')$. This composition is well-defined:

$$e'' \circ g \circ f \circ e = e'' \circ (e'' \circ g \circ e') \circ (e' \circ f \circ e) \circ e = (e'' \circ g \circ e') \circ (e' \circ f \circ e) = g \circ f.$$

Note that the identity map on (X, e) is e .

The pseudo-abelian envelope construction comes with a fully faithful functor $C \rightarrow \text{kar}(C)$ that sends any object X to (X, id_X) and any morphism to itself. The functor $C \rightarrow \text{kar}(C)$ is initial among functors from C to pseudo-abelian categories.

Given an idempotent $e : X \rightarrow X$ in C , its image in $\text{kar}(C)$ splits:

$$\begin{aligned} (X, \text{id}_X) &\xrightarrow{e} (X, \text{id}_X) = (X, \text{id}_X) \xrightarrow{e} (X, e) \xrightarrow{e} (X, \text{id}_X) \\ (X, e) &\xrightarrow{e} (X, \text{id}_X) \xrightarrow{e} (X, e) = (X, e) \xrightarrow{e} (X, e) = \text{id}_{(X, e)} \end{aligned}$$

Definition 1.1.13. The category of (*pure*) *effective motives*, $M_{\sim}^{\text{eff}}(k)_R$, is defined by taking the pseudo-abelian envelope of $C_{\sim}V(k)_R^0$. The objects of $M_{\sim}^{\text{eff}}(k)_R$ are pairs (X, e) where $X \in V(k)$ and e is an element of $\mathcal{Z}_{\sim}^{\dim X}(X \times X)$ that is idempotent with respect to the operation of composition we defined on cycles.

We then have a fully faithful functor

$$C_{\sim}V(k)_R^0 \rightarrow M_{\sim}^{\text{eff}}(k)_R \quad (1.3)$$

sending any variety X in $\text{ob } C_{\sim}V(k)_R$ to (X, id) .

Remark 1.1.14. The composition of (1.2) with (1.3) is called

$$\mathfrak{h}_{\sim, R}^{\text{eff}} : V(k)^{\text{op}} \rightarrow M_{\sim}^{\text{eff}}(k)_R.$$

It maps any variety X to (X, id) and any morphism of varieties $f : X \rightarrow Y$ to $f^* \in \mathcal{Z}_{\sim}^{\dim X}(Y \times X)_R$.

Definition 1.1.15. Fix a ring R and an adequate equivalence relation \sim on R -cycles. Given any smooth, projective variety X , $\mathfrak{h}_{\sim, R}^{\text{eff}}(X)$ is its *effective motive* with R -coefficients modulo \sim .

Definition 1.1.16. Fix coefficient ring R and an adequate equivalence relation \sim on R -cycles. The category of pure motives $M_{\sim}(k)_R$ has objects triples (X, e, n) where $X \in V(k)$, e is an idempotent and n is an integer.

In the literature, the motive $(\text{Spec } k, \text{id}_{\text{Spec } k}, 1)$ is referred to as the Tate motive, whereas its dual $(\text{Spec } k, \text{id}_{\text{Spec } k}, -1)$ is denoted by \mathbb{L} and called the Lefschetz motive. Morphisms $(X, e, n) \rightarrow (Y, e', n')$ are in correspondence with elements of $e \circ \mathcal{Z}_{\sim}^{\dim X - n + n'}(X \times Y)_R \circ e'$.

Remark 1.1.17. The category of pure motives can be thought of as the category of effective motives with the addition of Tate twists; the object (X, e, n) denotes (X, e) with n Tate twists. See Section 1.2 for further discussion of Tate twists and their meaning in different Weil cohomology theories.

Definition 1.1.18. There is a fully faithful functor

$$M_{\sim}^{\text{eff}}(k)_R \rightarrow M_{\sim}(k)_R$$

mapping objects $(X, e) \in \text{ob } M_{\sim}^{\text{eff}}(k)_R$ to $(X, e, 0)$. We call its composition with $\mathfrak{h}_{\sim, R}^{\text{eff}} : V(k)^{\text{op}} \rightarrow M_{\sim}^{\text{eff}}(k)_R$

$$\mathfrak{h}_{\sim, R} : V(k)^{\text{op}} \rightarrow M_{\sim}(k)_R. \quad (1.4)$$

For any $X \in \text{ob } V(k)$, $\mathfrak{h}_{\sim, R}(X)$ is the motive of X with R -coefficients modulo \sim .

Remark 1.1.19. Since the functor $M_{\sim}^{\text{eff}}(k)_R \rightarrow M_{\sim}(k)_R$ is fully faithful, given varieties X and Y , showing that $\mathfrak{h}_{\sim, R}^{\text{eff}}(X) \cong \mathfrak{h}_{\sim, R}^{\text{eff}}(Y)$ is equivalent to showing that $\mathfrak{h}_{\sim, R}(X) \cong \mathfrak{h}_{\sim, R}(Y)$.

Definition 1.1.20. For any fixed coefficient ring R , the objects of the category $M_{\sim_{\text{rat}}}(k)_R$ constructed with the rational equivalence relation are called *Chow motives* with coefficients in R . When \sim is given by homological equivalence, the objects of $M_{\sim_{\text{hom}}}(k)_R$ are sometimes called *Grothendieck motives* with coefficients in R . When \sim is given by numerical equivalence, the objects of $M_{\sim}(k)_R$ are called *numerical motives* with coefficients in R .

So, the category of Chow motives with rational coefficients, which we will see several times in the remainder of the chapter, is $\mathfrak{h}_{\sim_{\text{rat}}, \mathbb{Q}}^{\text{eff}}$, and the Chow motive (with rational coefficients) of a smooth, projective variety X is $\mathfrak{h}_{\sim_{\text{rat}}, \mathbb{Q}}(X)$.

Tensor structure

Categories of pure motives have tensor structures, which we introduce in this section. First we give some preliminary terminology.

Definition 1.1.21. Given two categories C and D , the *product category* $C \times D$ has objects given by pairs (X, Y) of objects $X \in C$ and $Y \in D$, and morphisms $(X, Y) \rightarrow (X', Y')$ are likewise given by pairs of morphisms $X \rightarrow X'$ in C and $Y \rightarrow Y'$ in D . Composition is given by the respective composition of the first entries in C and the second entries in D : $(f_1, f_2) \circ (g_1, g_2) = (f_1 \circ g_1, f_2 \circ g_2)$.

A *bifunctor* is a functor whose domain is a product category.

Definition 1.1.22. A category \mathcal{T} is a \otimes -category over a commutative ring R if its sets of morphisms have an R -module structure, and there is a tensor operation \otimes satisfying the following conditions (André [3, 2.2.2]):

- (i) The operation \otimes is given by a bilinear bifunctor $\otimes : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$. The functor \otimes is *bilinear* if its maps from sets of morphisms in $\mathcal{T} \times \mathcal{T}$ to sets of morphisms in \mathcal{T} are bilinear maps.
- (ii) There is a unital object $\mathbb{1}$.
- (iii) For any objects $L, M, N \in \mathcal{T}$, there are functorial isomorphisms $a_{LMN} : (L \otimes M) \otimes N \cong L \otimes (M \otimes N)$, $c_{MN} : M \otimes N \cong N \otimes M$, such that $c_{NM} = c_{MN}^{-1}$, and $u_m : M \otimes \mathbb{1} \cong M$, $u'_m : \mathbb{1} \otimes M \cong M$ giving the associativity of \otimes , commutativity of \otimes , and left and right tensor with the unital object. These functorial isomorphisms are not equalities, and so their interaction with each other is not immediate and must be defined by axioms called coherence conditions. We will not use coherence conditions explicitly here and so will not include a definition; see Chapitre I Section 1 of Saavedra Rivano [39] for further details.

Definition 1.1.23. A \otimes -category \mathcal{T} has a *rigid* \otimes -category structure if it satisfies the following additional condition:

- (iv) There is an autoduality ${}^\vee : \mathcal{T} \rightarrow \mathcal{T}^{\text{op}}$ such that for any $M \in \text{ob } \mathcal{T}$, the functor $(-) \otimes M^\vee$ is left adjoint to $(-) \otimes M$ and $M^\vee \otimes (-)$.

Definition 1.1.24. A functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ between \otimes -categories over R is a \otimes -functor if it satisfies the following compatibility conditions with the \otimes -category structures on \mathcal{T} and \mathcal{T}' :

1. F is R -linear, meaning that maps on hom-sets induced by F are R -module maps,
2. F is compatible with the tensor and unit operations: for any $M, N \in \text{ob } \mathcal{T}$, $F(M \otimes N) = FM \otimes FN$ and $F\mathbb{1}_{\mathcal{T}} = \mathbb{1}_{\mathcal{T}'}$.

Remark 1.1.25. If a functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ is a \otimes -functor between \otimes -categories, then if \mathcal{T} and \mathcal{T}' are rigid, F is automatically compatible with the duality operation.

Example 1.1.26. The category of varieties $V(k)$ has a \otimes -category structure given by the fiber product \times_k with $\text{Spec}(k)$ as the unital object.

For any ring of coefficients R and adequate equivalence relation \sim on R -cycles, $C_\sim V(k)_R$ inherits a \otimes -structure from $V(k)$ via the functor $V(k)^{\text{op}} \rightarrow C_\sim V(k)_R$ (1.1) since the categories have the same sets of objects.

Furthermore, the category of motives $M_\sim(k)_R$ has a \otimes -structure: for any (X, e, r) , $(Y, e', r') \in \text{ob } M_\sim(k)_R$, the tensor product is $(X, e, r) \otimes (Y, e', r') = (X \times_k Y, e \otimes e', r + r')$ and the unital object is $(\text{Spec } k, \text{id}_{\text{Spec } k}, 0)$, the image of $\text{Spec } k$ under the functor $\mathfrak{h}_{\sim, R}$ (1.4).

Example 1.1.27. $M_\sim(k)_R$ is a rigid \otimes -category: it has an autoduality given by $(X, e, r)^\vee = (X, e^t, \dim X - r)$, where e^t denotes the transpose of e . Note that we need the Tate twists here to achieve this rigid \otimes -category structure since the sub- \otimes -category of effective motives $M_\sim^{\text{eff}}(k)_R$ inside $M_\sim(k)_R$ is not stable under the duality operation.

1.2 Weil cohomology theories

So far we have defined categories of motives by constructing them; however, the category of Chow motives with \mathbb{Q} -coefficients enjoys a universal property of relating to Weil cohomology theories (see Section 1.2). In this section we define a Weil cohomology theory, discuss realizations of Tate twists in different Weil cohomologies, define Chern and Todd classes, which we will need in later chapters, and discuss the universal property of Chow motives with rational coefficients.

Weil cohomology theories are given by functors $H^* : V(k)^{\text{op}} \rightarrow \text{VecGr}(K)$ from the category of smooth, projective varieties over a (not necessarily algebraically closed) field k to the category of \mathbb{Z} -graded K -vector spaces for some choice of characteristic 0 field K , called the coefficient field of H^* , along with some other data and axioms, including Tate twists. Note that not every definition of a Weil cohomology includes the Tate twist: it is left out of the Kleiman [24] as it is unnecessary for the purpose of that article. However, it is important to keep track of Tate twists in this thesis, as we will see in Chapter 3.

Definition 1.2.1. A Weil cohomology is given by the following data and axioms. This definition closely follows that given in de Jong's note [15]; compare also the definition of a Poincaré duality theory with supports, as given by Bloch and Ogus in [7]:

- (D1) For every smooth, projective algebraic variety X over k , a \mathbb{Z} -graded commutative algebra $H^*(X)$ over K with a K -bilinear multiplication $H^*(X) \times H^*(X) \rightarrow H^*(X)$ called the cup product. By graded commutative we mean that $\alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha$ for homogeneous $\alpha, \beta \in H^*(X)$.

- (D2) For every morphism $f : X \rightarrow Y$ of smooth projective varieties over K , a pullback map $f^* : H^*(Y) \rightarrow H^*(X)$ preserving grading. If we think of Weil cohomology as a functor H^* , then $f^* = H^*(f)$.
- (D3) A 1-dimensional K -vector space $K(1)$, which gives rise to Tate twists: for any K -vector space V , $V(n) := V \otimes K(1)^{\otimes n}$. If n is negative, $V(n) := V \otimes \text{Hom}(K(1)^{\otimes -n}, K)$.
- (D4) For every smooth projective variety X over k , a trace map $\text{Tr} : H^{2\dim X}(X)(\dim X) \rightarrow K$.
- (D5) For every smooth projective variety X over k and every closed subvariety $Z \subset X$ of codimension c , there is a cohomology class $\text{cl}(Z) \in H^{2c}(X)(c)$.

A Weil cohomology theory must then satisfy the following axioms:

- (A1) Each $H^i(X)$ is a finite-dimensional K -vector space.
- (A2) If $H^i(X) \neq (0)$, then $i \in [0, 2\dim(X)]$
- (A3) Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ of nonsingular projective varieties, we have $(g \circ f)^* = f^* \circ g^*$, which is equivalent to the functoriality of H^* .
- (A4) Given nonsingular projective varieties X, Y , consider the pullbacks of the standard projection maps $p_X^* : H^*(X) \rightarrow H^*(X \times Y)$ and $p_Y^* : H^*(Y) \rightarrow H^*(X \times Y)$. The map

$$\begin{aligned} H^*(X) \otimes_K H^*(Y) &\rightarrow H^*(X \times Y) \\ \alpha \otimes \beta &\mapsto p_X^*(\alpha) \cup p_Y^*(\beta) \end{aligned}$$

is an isomorphism of vector spaces, known as the Künneth isomorphism.

- (A5) (Poincaré duality) For every nonsingular projective variety X and integer $0 \leq j \leq 2\dim X$, the cup product

$$H^j(X) \times H^{2\dim X - j}(X)(\dim X) \xrightarrow{\cup} H^{2\dim X}(X)(\dim X) \xrightarrow{\text{Tr}} K$$

induces a perfect duality between $H^j(X)$ and $H^{2\dim X - j}(X)(\dim X)$.

- (A6) Trace maps are compatible with products: given nonsingular projective varieties X, Y and any $\alpha \in H^{2\dim X}(X)(\dim X)$, $\beta \in H^{2\dim Y}(Y)(\dim Y)$, the trace map

$$\text{Tr}_{X \times Y} : H^{2\dim X + 2\dim Y}(X \times Y)(\dim X + \dim Y) \rightarrow K$$

satisfies $\text{Tr}_{X \times Y}(p_X^*(\alpha) \cup p_Y^*(\beta)) = \text{Tr}_X(\alpha) \text{Tr}_Y(\beta)$.

- (A7) Given nonsingular projective varieties X, Y and closed subvarieties $Z \subset X$, $W \subset Y$, we have $\text{cl}(Z \times W) = p_X^*(\text{cl}(Z)) \cup p_Y^*(\text{cl}(W))$.

(A8) Let $f : X \rightarrow Y$ be a morphism of nonsingular projective varieties and $Z \subset X$ be a closed subvariety. Note that $f_*[Z] = m[f(Z)]$ where m is the degree of the morphism $Z \rightarrow f(Z)$. Then, $\text{Tr}_X(\text{cl}(Z) \cup f^*\alpha) = m \text{Tr}_Y(\text{cl}(f(Z)) \cup \alpha)$ for every $\alpha \in H^{2\dim Z}(Y)(\dim Z)$.

(A9) Suppose that $f : X \rightarrow Y$ is a morphism of nonsingular projective varieties, and let $Z \subset Y$ be a closed subvariety. Assume $\dim f^{-1}(Z) = \dim Z + \dim X - \dim Y$. Write the cycle associated to $f^{-1}(Z)$ as $[f^{-1}(Z)]_k = \sum n_i Z_i$ where $k = \dim Z + \dim X - \dim Y$. Then

$$f^* \text{cl}(Z) = \sum n_i \text{cl}(Z_i).$$

(A10) Let $x = \text{Spec } k$. Then $\text{Tr}_x(\text{cl}(x)) = 1$.

Examples of Weil cohomologies include singular cohomology, de Rham cohomology, crystalline cohomology and – most important for the work here over finite fields – ℓ -adic étale cohomology.

Note that axiom (A8) refers to the notion pushforward, but only between cycles. We define pushforward acting on Weil cohomologies as in [15]:

Definition 1.2.2. Given a morphism $f : X \rightarrow Y$ between nonsingular, projective varieties, its pushforward $f_* : H^*(X) \rightarrow H^{*-2r}(Y)(-r)$, where $r = \dim X - \dim Y$, sends any $\alpha \in H^j(X)$ to the unique element $f_*(\alpha)$ of $H^{j-2r}(Y)(-r)$ such that $\text{Tr}_Y(f_*(\alpha) \cup \beta) = \text{Tr}_X(\alpha \cup f^*(\beta))$ for all $\beta \in H^{2\dim X-j}(Y)(\dim X)$. Equivalently, we can induce the pushforward map by using Poincaré duality (A5) and the dual of the pullback map, as follows:

$$\begin{aligned} H^j(X) &\xrightarrow{(A5)} H^{2\dim X-j}(X)(\dim X)^\vee \xrightarrow{(f^*)^\vee} H^{2\dim X-j}(Y)(\dim X)^\vee \\ &\xrightarrow{(A5)} H^{2\dim Y-2\dim X+j}(Y)(\dim Y - \dim X) = H^{j-2r}(Y)(-r) \end{aligned}$$

1.2.3 Tate twists and the cohomology of the projective line. In some treatments of Weil cohomologies, the Tate twist is defined by the relationship $H^2(\mathbb{P}^1) \cong K(-1)$. We can use our definition of a Weil cohomology theory to construct such an isomorphism: The trace map $\text{Tr} : H^2(\mathbb{P}^1)(1) \rightarrow K$ is nonzero since, by axiom (A10), it maps the class of a closed point in \mathbb{P}^1 to 1. The Poincaré duality axiom (A5) implies that both the 0-degree and top degree cohomology groups must be 1-dimensional. Twisting both sides of the trace map by -1 then yields an isomorphism $H^2(\mathbb{P}^1) \cong K(-1)$.

Using this information, we interpret the realization of Tate twists in each of the classical Weil cohomology theories as follows (see [3, 3.4.4]):

1.2.4 Realizations of Tate twists. In ℓ -adic étale cohomology, a Tate twist corresponds to a twist by the p -adic cyclotomic character: Put another way, $H^0(\mathbb{P}^1, \mathbb{Q}_\ell)$ and $H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$ are both isomorphic to \mathbb{Q}_ℓ , but $H^0(\mathbb{P}^1, \mathbb{Q}_\ell)$ carries a trivial Galois action and $H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$ is dual to $\mathbb{Q}_\ell(1) := \varprojlim_m \mu_{\ell^m}$, which has the (nontrivial) natural cyclotomic Galois action of $\text{Gal}(\bar{k}/k)$. So, a Tate twist by r in this situation corresponds to having the action of the r^{th} power of the cyclotomic character.

In de Rham cohomology, the Tate twist tracks the Hodge filtration: $H^2(\mathbb{P}^1)$ has the Hodge filtration of $F^{\leq 0} = 0$ and $F^{> 0} = k$. The effect of a Tate twist by r is then to shift the Hodge filtration by $-r$.

In singular cohomology, $H^2(\mathbb{P}^1) = \frac{1}{2\pi i} \mathbb{Q}$, and the bigrading of $H^2(\mathbb{P}^1) \otimes \mathbb{C} \cong \mathbb{C}$ is purely of type $(1, 1)$. So a twist by r in singular cohomology is a multiplication by $(2\pi i)^r$ with a scaling by $(-r, -r)$ in the Hodge decomposition.

Finally, crystalline cohomology can be defined when the base field k is perfect of positive characteristic p . The coefficient field K is defined to be $W(k)[1/p]$, the field of fractions of the ring of Witt vectors $W(k)$ over k . It comes with a Frobenius map ϕ . The cohomology group $H^2(\mathbb{P}^1) = W(k)[1/p]$, but the presence of the twist changes the Frobenius action to $p \cdot \phi$. More generally, a Tate twist of r on crystalline cohomology is given by multiplying the crystalline Frobenius map by p^{-r} .

Chern classes and characters, and Todd classes

In later chapters, we will need to use Chern and Todd classes, so we give a construction of them here.

Definition 1.2.5. In [15, Exercise 19] de Jong defines, for any smooth, projective variety X , a map $c_1 : \text{Pic}(X) \rightarrow H^2(X)(1)$ giving the *first Chern classes* of line bundles: Given a line bundle \mathcal{L} on X , if it has a nonzero section so that $\mathcal{L} \cong \mathcal{O}(D)$ where $D = \text{div}(s)$, then we set $c_1(\mathcal{L}) = \text{cl}(D)$. Then in general, we can write any line bundle \mathcal{L} as $\mathcal{O}(D_1 - D_2)$ where D_1 and D_2 are effective, and set $c_1(\mathcal{L}) = \text{cl}(D_1) - \text{cl}(D_2)$. The map c_1 is functorial: for any morphism of varieties $f : X \rightarrow Y$, $f^*c_1(\mathcal{L}) = c_1(f^*\mathcal{L})$.

Definition 1.2.6. The *total Chern class* $c(\mathcal{E})$ of any vector bundle \mathcal{E} on X is a sum $\sum_i c_i(\mathcal{E})$ of elements $c_i(\mathcal{E}) \in H^{2i}(X)(i)$, called the Chern classes, and is uniquely characterized by the following properties [15, Exercise 21]:

1. If the vector bundle \mathcal{E} is of rank 1, then $c(\mathcal{E}) = 1 + c_1(\mathcal{E})$.
2. Total Chern classes are functorial with respect to pullbacks: for any morphism of varieties $f : X \rightarrow Y$, $f^*c(\mathcal{E}) = c(f^*\mathcal{E})$.
3. For any short exact sequence of vector bundles $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$, the total Chern classes of the \mathcal{E}_i are related by the equality $c(\mathcal{E}_2) = c(\mathcal{E}_1) \cup c(\mathcal{E}_3)$.

Remark 1.2.7. One could define intersection-theoretic total Chern classes (and the other objects we will define in terms of them in this section) inside the Chow group of X modulo rational equivalence, and recover the elements in cohomology groups defined above by applying the cycle class map. Chern classes in the Chow group are defined by the same set of axioms as shown above, except each $c_i(\mathcal{E})$ is contained in $\mathcal{Z}_{\sim_{\text{rat}}}^i(X)_{\mathbb{Q}}$. See also [20, Chapter 3] for further discussion.

Definition 1.2.8. The *Chern polynomial* of a vector bundle \mathcal{E} on X is the following polynomial in the formal variable t :

$$c_t(\mathcal{E}) = \sum_i c_i(\mathcal{E}) t^i.$$

We formally factor the Chern polynomial as $c_t(\mathcal{E}) = \prod_i (1 + \alpha_i t)$, and call the roots α_i the *Chern roots*.

Definition 1.2.9. The *Chern character* is the sum of the exponentials of the Chern roots:

$$\text{ch}(\mathcal{E}) = \sum_i \exp(\alpha_i) \quad \text{where} \quad \exp(\alpha_i) := \sum \frac{\alpha_i^n}{n!}.$$

From the definition of the Chern character, we can deduce that for any short exact sequence of vector bundles $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$,

$$\text{ch}(\mathcal{E}_2) = \text{ch}(\mathcal{E}_1) + \text{ch}(\mathcal{E}_3), \tag{1.5}$$

and for any vector bundles \mathcal{E} and \mathcal{E}' , $\text{ch}(\mathcal{E} \otimes \mathcal{E}') = \text{ch}(\mathcal{E}) \cdot \text{ch}(\mathcal{E}')$.

Definition 1.2.10. The *Todd class* $\text{td}(\mathcal{E})$ of a vector bundle \mathcal{E} on a smooth, projective variety is also defined in terms of its Chern roots:

$$\text{td}(\mathcal{E}) := \prod_i Q(\alpha_i) \quad \text{where} \quad Q(\alpha) := \frac{\alpha}{1 - e^{-\alpha}}$$

Given a short exact sequence of vector bundles as above

$$\text{td}(\mathcal{E}_2) = \text{td}(\mathcal{E}_1) \cdot \text{td}(\mathcal{E}_3). \tag{1.6}$$

Remark 1.2.11. So far we have only defined the Chern characters and Todd classes of vector bundles. If \mathcal{F} is a coherent sheaf on a smooth, projective variety X , we define $\text{ch}(\mathcal{F})$ and $\text{td}(\mathcal{F})$ by taking finite resolution of \mathcal{F} by vector bundles and applying (1.5) and (1.6).

Motives as classifiers of Weil cohomology groups

Earlier, we defined motives from a constructive point of view. Here we discuss the universal property of Chow motives and the role of Weil cohomologies in this universal property.

Proposition 1.2.12 (André [3, 4.2.4]). *Let \mathcal{T} be a rigid \otimes -category with coefficients in R that is pseudo-abelian and contains an object \mathbb{L} that is invertible with respect to the tensor product.*

Suppose we have the following:

- (i) *There is a monoidal functor $H : V(k)^{\text{op}} \rightarrow \mathcal{T}$ such that the maps $\mathbb{P}^1 \rightarrow \text{Spec } k = \{\infty\} \rightarrow \mathbb{P}^1$ induce a decomposition $H(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$.*

(ii) For all $X \in V(k)$ purely of dimension d , there is a map $\mathrm{tr}_X : H(X) \rightarrow \mathbb{L}^{\otimes d}$ such that $\mathrm{tr}_{X \times Y} = \mathrm{tr}_X \otimes \mathrm{tr}_Y$, and that identifies $H(X)^\vee$ with $H(X) \otimes \mathbb{L}^{\otimes -d}$.

(iii) For any $X \in V(k)$, there are R -linear homomorphisms

$$c_X^r : \mathcal{Z}_{\sim \mathrm{rat}}^r(X)_R \rightarrow \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, H(X) \otimes \mathbb{L}^{\otimes -r})$$

contravariant in X , such that $c_{X \times Y}^n = \sum_{r+s=n} c_X^r \otimes c_Y^s$, and the composite

$$\mathcal{Z}_{\sim \mathrm{rat}}^d(X)_R \xrightarrow{c_X^d} \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, H(X) \otimes \mathbb{L}^{\otimes -d}) \xrightarrow{\mathrm{tr}_X \otimes \mathrm{id}_{\mathbb{L}^{\otimes -d}} \circ (-)} \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathbb{1})$$

coincides with the degree map of 0-cycles when X is of pure dimension d .

Then, H admits a unique factorization

$$V(k)^{\mathrm{op}} \xrightarrow{\mathfrak{h}} M_{\sim \mathrm{rat}}(k)_R \xrightarrow[\mathcal{H}]{\exists! \omega_H} \mathcal{T}$$

such that $\mathbb{L} = \omega_H(\mathbb{1}(-1))$, $\omega_H \circ \mathrm{Tr}_X = \mathrm{tr}_X \circ \omega_H$ and c_X^r is given by the map

$$\mathcal{Z}_{\sim \mathrm{rat}}^r(X)_R = \mathrm{Hom}_{M_{\sim \mathrm{rat}}(k)_R}(\mathbb{1}, (X, \mathrm{id}, r) \rightarrow \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, H(X) \otimes \mathbb{L}^{\otimes -r}))$$

induced by ω_H . The quadruple

$$(M_{\sim \mathrm{rat}}(k)_R, \mathbb{1}(-1), \mathrm{Tr}_X, \{\gamma_X^r\}_r)$$

(where γ_X^r gives the identification between cycles in $\mathcal{Z}_{\sim \mathrm{rat}}^r(X)_R$ and the maps

$$\mathrm{Hom}_{M_{\sim \mathrm{rat}}(k)_R}(\mathbb{1} = (\mathrm{Spec} k, \mathrm{id}, 0), H(X) \otimes \mathbb{L}^{\otimes -r})$$

from the definition of maps in $M_{\sim \mathrm{rat}}(k)_R$) is then universal among quadruples

$$(\mathcal{T}, \mathbb{L}, \mathrm{tr}_X, \{c_X^r\}_r)$$

satisfying properties as articulated in the above conditions.

Remark 1.2.13. Any Weil cohomology theory with coefficients field K containing the ring R satisfies conditions (i), (ii), (iii). For (i), recall that for Weil cohomology theories, we identify $\mathbb{1}$ with $K \cong H^0(\mathbb{P}^1)$ and \mathbb{L} with $H^2(\mathbb{P}^1)$. Pulling back by the maps in condition (i) induces a decomposition of $H^*(\mathbb{P}^1)$ into $H^0(\mathbb{P}^1)$ and the rest of the cohomology of \mathbb{P}^1 , so all that remains is to prove that the only nonzero cohomology groups of \mathbb{P}^1 are in degrees 0 and 2. The group $H^1(\mathbb{P}^1)$ can be shown to vanish by computing the self-intersection number of the diagonal on $\mathbb{P}^1 \times \mathbb{P}^1$ and using the Lefschetz fixed-point theorem. The trace map in (D4) induces the map in (ii). The maps in (iii) come from the maps in (D5). In the case of a Weil cohomology theory, we are mapping to the category of graded vector spaces over a field K , so here, $\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, H(X) \otimes \mathbb{L}^{\otimes -r}) = \mathrm{Hom}_{\mathrm{VectGr}}(K, H^*(X)(r))$.

By Proposition 1.2.12 and Remark 1.2.13, given the data of a Weil cohomology theory, we can produce a functor into it from a category of Chow motives with rational coefficients. The following result shows when a functor with Chow motives as its domain determines a Weil cohomology theory:

Proposition 1.2.14 ([3, 4.2.5.1]). *Giving a Weil cohomology theory with coefficients in a field K containing a ring R is equivalent to a \otimes -functor*

$$H^* : M_{\sim_{\text{rat}}}(k)_R \rightarrow \text{VectGr}_K$$

such that $H^i(K(-1)) = 0$ for all $i \neq 2$.

As we would expect from this discussion, Weil cohomology theories share some qualities. For example, for any variety X , the dimension of $H^i(X)$ is constant as H ranges over classical Weil cohomology theories [3, 4.2.5.2].

1.3 Lefschetz fixed-point and zeta functions

For any Weil cohomology theory H , the Lefschetz fixed-point theorem holds (see Proposition 1.3.6 and Section 4 of Kleiman [24]):

Theorem 1.3.1. *Let X be a nonsingular projective variety and $f : X \rightarrow X$ an endomorphism. Then*

$$(\Gamma_f \cdot \Delta) = \sum (-1)^i \text{Tr}(f^* | H^i(X, \mathbb{Q}_\ell))$$

where Γ_f is the graph of f , Δ is the diagonal of X and $(\Gamma_f \cdot \Delta)$ denotes their intersection number.

Definition 1.3.2. For each positive integer m , fix a field \mathbb{F}_{q^m} with q^m elements so that $\mathbb{F}_{q^{m+1}}$ is an extension of \mathbb{F}_{q^m} . The (Hasse–Weil) *zeta function* of a nonsingular projective variety X over a finite field \mathbb{F}_q is the following power series in t (see for instance Milne [29]):

$$\zeta(X) = \exp\left(\sum_{m \geq 1} N_m(X) \frac{t^m}{m}\right),$$

where $N_m(X)$ is the number of points of X with coordinates in \mathbb{F}_{q^m} .

Recall the following classical theorem:

Theorem 1.3.3 (Fermat’s little theorem). *Let q be a prime. If a is an integer not divisible by q , then $a^{q^n} \cong a \pmod{q^n}$.*

Remark 1.3.4. Fermat's little theorem suggests that we might hope to count points using the Lefschetz fixed point theorem: If we have a Frobenius map φ that sends points x in a variety X (actually in $\bar{X} := X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$; see below) to their m^{th} powers, then $(\Gamma_{\varphi^m} \cdot \Delta) = N_m(X)$. For some intuition we can check this in the case when the intersection of Γ_{φ^m} and the diagonal Δ is transverse: then $\Gamma_{\varphi^m} \cdot \Delta$ is precisely all pairs of points both of the form (x, x) and $(x, \varphi^m(x))$, and so corresponds to exactly the points $x = \varphi^m(x)$.

However, how should we correctly think of the action of Frobenius on a scheme?

Frobenius

In this section we define several Frobenius morphisms on schemes see Section 1 of Houzel's Exposé XV in SGA V [42] further discussion of Frobenius morphisms.

Definition 1.3.5. For a scheme X/\mathbb{F}_q , the first notion of Frobenius endomorphism we might guess at is the *absolute Frobenius* morphism $\text{Fr}_X : X \rightarrow X$, which acts on the structure sheaf by raising its sections to their q^{th} powers. The absolute Frobenius morphism is functorial on schemes over \mathbb{F}_q ; for any map of \mathbb{F}_q -schemes $f : X \rightarrow Y$, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\text{Fr}_X} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{\text{Fr}_Y} & Y \end{array} \quad (1.7)$$

Example 1.3.6. Consider the action of the absolute Frobenius morphism on $\text{Spec}(\mathbb{F}_q[t])$. The underlying map on rings $\mathbb{F}_q[t] \rightarrow \mathbb{F}_q[t]$ maps $t - a$, for any $a \in \mathbb{F}_q$, to $(t - a)^q = t^q - a^q$. The preimage of the prime ideal $(t - a)$ is $(t - a)$.

Since absolute Frobenius morphisms act identically on the underlying topological spaces of schemes, we cannot use them to calculate zeta functions.

To construct a Frobenius morphism that we can use to calculate zeta functions, we need to pass to $\bar{X} := X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$, the base change of X/\mathbb{F}_q over the algebraic closure of \mathbb{F}_q .

The following results give us information about the behavior of ℓ -adic étale cohomology under base change (see also Freitag and Kiehl [18, Theorem 6.1]):

Theorem 1.3.7 (Proper base change, [29, Theorem 17.7]). *Let $\pi : X \rightarrow S$ be a proper morphism, and let \mathcal{F} be a constructible sheaf on X . Then $R^r \pi_* \mathcal{F}$ is constructible for all $r \geq 0$, and for every geometric point $\bar{s} \rightarrow S$, $(R^r \pi_* \mathcal{F})_{\bar{s}} = H^r(X_{\bar{s}}, \mathcal{F}|_{X_{\bar{s}}})$, where $X_{\bar{s}} := X \times_S \bar{s}$ and $\mathcal{F}|_{X_{\bar{s}}}$ is the inverse image of \mathcal{F} under the map $X_{\bar{s}} \rightarrow X$.*

Corollary 1.3.8 ([29, Corollary 17.8(b)]). *Let X be a complete variety over a separably closed field k and let \mathcal{F} be a constructible sheaf on X . For any separably closed field $k' \supset k$, $H^r(X, \mathcal{F}) = H^r(X', \mathcal{F}')$, where X' is the base change of X to k' and \mathcal{F}' is the inverse of image of \mathcal{F} under the map $X' \rightarrow X$.*

Remark 1.3.9. Given a smooth projective variety X over \mathbb{F}_q , since \mathbb{F}_q is separably closed, as is $\overline{\mathbb{F}}_q$, and \mathbb{Q}_ℓ is locally constant, hence constructible, Corollary 1.3.8 implies $H^i(X, \mathbb{Q}_\ell) \cong H^i(\overline{X}, \mathbb{Q}_\ell)$, and so $\mathrm{Tr}(\varphi^*|H^i(X, \mathbb{Q}_\ell)) \cong \mathrm{Tr}(\varphi^*|H^i(\overline{X}, \mathbb{Q}_\ell))$.

Definition 1.3.10. Let S be a scheme of characteristic p and suppose X is an S -scheme via the morphism $f : X \rightarrow S$. Then the Frobenius map of X *relative to* S , $\mathrm{Fr}_{X/S}$ is the base change of the absolute Frobenius Fr_S to X . The fiber product of Fr_S and f is denoted by $X^{(p/S)}$, or $X^{(p)}$, when S is understood:

$$\begin{array}{ccc} X^{(p/S)} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ S & \xrightarrow{\mathrm{Fr}_S} & S \end{array}$$

Remark 1.3.11. Since absolute Frobenius is functorial in schemes over \mathbb{F}_q , the diagram (1.7) factors through $X^{(p/S)}$ as follows:

$$\begin{array}{ccccc} X & & & & \\ & \searrow \mathrm{Fr}_X & & \searrow \mathrm{Fr}_X & \\ & \searrow \mathrm{Fr}_{X/S} & & \searrow \mathrm{Fr}_{X/S} & \\ & & X^{(p/S)} & \longrightarrow & X \\ & \searrow f & \downarrow & & \downarrow f \\ & & S & \xrightarrow{\mathrm{Fr}_S} & S \end{array}$$

The relative (to S) Frobenius is functorial on S -schemes. Also, $\mathrm{Fr}_{X/S}$ is an isomorphism if and only if f is étale [42, Exposé XV, §1, Proposition 2(c2), page 446].

Remark 1.3.12. Let X be a scheme over \mathbb{F}_q and $\overline{X} := X \otimes \overline{\mathbb{F}}_q$. In this case, $\overline{X}^{(p/X)} \cong \overline{X}$.

Example 1.3.13. If $X = \mathrm{Spec}(\mathbb{F}_q[t])$, $\overline{X}^{(p/X)}$ is Spec of the pushout of the following diagram:

$$\begin{array}{ccc} & & \overline{\mathbb{F}}_q[t] \\ & & \uparrow \\ \mathbb{F}_q[t] & \xleftarrow{(-)^q} & \mathbb{F}_q[t] \end{array}$$

The pushout is $\overline{\mathbb{F}}_q[t]$, where the maps $\mathbb{F}_q[t] \rightarrow \overline{\mathbb{F}}_q[t]$ and $\overline{\mathbb{F}}_q[t] \rightarrow \overline{\mathbb{F}}_q[t]$ filling in the diagram are given by inclusion and raising t to the q^{th} power but fixing elements of $\overline{\mathbb{F}}_q$, respectively. Then the map $\mathrm{Fr}_{\overline{X}/X} : \mathrm{Spec}(\overline{\mathbb{F}}_q[t]) \rightarrow \mathrm{Spec}(\overline{\mathbb{F}}_q[t])$ is induced by the map $\overline{\mathbb{F}}_q[t] \rightarrow \overline{\mathbb{F}}_q[t]$ that fixes t and maps elements of $\overline{\mathbb{F}}_q$ to their q^{th} powers; given any $a \in \overline{\mathbb{F}}_q$, $(t - a) \mapsto (t^q - a)$. So, considering a as a point on the variety, it is mapped to its p^{th} power by $\mathrm{Fr}_{\overline{X}/X}$.

The relative Frobenius map $\mathrm{Fr}_{\overline{X}/X}$ gives an endomorphism on \overline{X} that we may use to compute $\zeta(X)$ as described in Remark 1.3.4.

We can also define a Frobenius map for counting rational points in another way: as the *geometric* Frobenius map, the inverse of the *arithmetic* Frobenius map. These constructions are often preferable since they are a generalization of the maps $\mathrm{Fr}_{\mathrm{Spec} \bar{\mathbb{F}}_q}$ and its inverse, which are members of the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

Definition 1.3.14. Given a scheme X/\mathbb{F}_q , arithmetic Frobenius $\mathrm{Fr}_{X/\mathbb{F}_q}^a$ is the pullback of the absolute Frobenius map $\mathrm{Fr}_{\mathrm{Spec}(\bar{\mathbb{F}}_q)} : \mathrm{Spec} \bar{\mathbb{F}}_q \rightarrow \mathrm{Spec} \bar{\mathbb{F}}_q$ to \bar{X} :

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\mathrm{Fr}_{X/\mathbb{F}_q}^a} & \bar{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec} \bar{\mathbb{F}}_q & \xrightarrow{\mathrm{Fr}_{\mathrm{Spec} \bar{\mathbb{F}}_q}} & \mathrm{Spec} \bar{\mathbb{F}}_q \end{array}$$

Example 1.3.15. If $X = \mathbb{F}_q[t]$, arithmetic Frobenius $\mathrm{Fr}_{X/\mathbb{F}_q}^a$ is induced by the map of the rings on the top of the following pushout diagram, which fixes t and sends elements of $\bar{\mathbb{F}}_q$ to their q^{th} powers:

$$\begin{array}{ccc} \bar{\mathbb{F}}_q[t] & \xleftarrow{\quad} & \bar{\mathbb{F}}_q[t] \\ \uparrow & & \uparrow \\ \bar{\mathbb{F}}_q & \xleftarrow{(-)^q} & \bar{\mathbb{F}}_q \end{array}$$

Definition 1.3.16. Since $\mathrm{Fr}_{\mathrm{Spec}(\bar{\mathbb{F}}_q)}$ is an isomorphism, arithmetic Frobenius is an isomorphism. So, we may take the inverse of arithmetic Frobenius, which we call *geometric Frobenius*. We will denote the geometric Frobenius morphism acting on \bar{X} by φ_X , and suppress the subscript when it is clear from context which variety is being acted on.

Example 1.3.17. When $X = \mathbb{F}_q[t]$, geometric Frobenius is induced by the map of rings $\bar{\mathbb{F}}_q[t] \rightarrow \bar{\mathbb{F}}_q[t]$ that fixes t but sends elements of $\bar{\mathbb{F}}_q$ to their q^{th} roots, and so it takes points to their q^{th} powers.

Remark 1.3.18. Let X/\mathbb{F}_q be a smooth, projective variety. The points in X with coordinates in \mathbb{F}_{q^m} will be those fixed by the m^{th} power of the geometric Frobenius map φ_X , and so by the Lefschetz fixed-point theorem,

$$N_m(X) = (\Gamma_{\varphi^m} \cdot \Delta) = \sum (-1)^i \mathrm{Tr}(\varphi^{m*} | H^i(X)).$$

for any Weil cohomology H defined for varieties over finite fields (we will use ℓ -adic étale cohomology to calculate the zeta functions, but try to keep the setting for this work as general as possible).

Zeta functions are thus invariant for any Weil cohomology theory (defined over a finite field), and so they are invariants of Chow motives with rational coefficients.

From zeta functions to motives

In the previous section, we showed that the zeta function of any variety over a finite field can be calculated from its cohomology groups in any Weil cohomology theory, and hence is a motivic invariant.

However, the converse, starting with equality of zeta functions and recovering an isomorphism of motives, is conjectural.

Remark 1.3.19. To give an isomorphism of the motives of varieties X and Y , we need to produce a cycle class on $X \times Y$, and here we give a discussion of the production of such a cycle. In later chapters, we will give arguments demonstrating, in some cases, the equality of the zeta functions of two varieties as a consequence of a Fourier–Mukai equivalence between them. A Fourier–Mukai equivalence induces a map on cohomology groups controlled by a cycle on $X \times Y$ called the Mukai vector (see Definition 3.1.1). However, when we produce (conjecturally) a cycle here using the equality of two zeta functions, it by no means recovers this Mukai vector – in general the Mukai vector of a Fourier–Mukai equivalence between two varieties does not define a morphism between their Chow motives.

The Tate conjectures get us partway toward producing a cycle class under rational equivalence. Tate’s original articles [44], and the slightly earlier [45] are excellent references for them.

Notation 1.3.20. Let X be a smooth projective scheme X over \mathbb{F}_q and ℓ coprime with q . The Tate conjectures are general concerned with the maps

$$V^j(X) := \mathbb{Q}_\ell \otimes_{\mathbb{Q}} A^j(X) \rightarrow H^{2j}(\bar{X}, \mathbb{Q}_\ell)(j) \quad (1.8)$$

where $H^{2j}(\bar{X}, \mathbb{Q}_\ell)(j)$ is an ℓ -adic étale cohomology group and $A^j(X)$ denotes the \mathbb{Q} -span of the image of $\mathcal{Z}^j(X)$ in $H^{2j}(\bar{X}, \mathbb{Q}_\ell)(j)$ under the cycle map Definition 1.2.1(D5).

In [44], Tate lists the following conjectures (we use the same abbreviations for their names as those in that article):

Conjecture 1.3.21.

- $T^j(X)$ For j , (1.8) is surjective. The collection of these statements for all j is in general known as the Tate conjecture, although sometimes that name is reserved only for the case where $j = 1$.
- $I^j(X)$ For j , the map (1.8) is injective.
- $E^j(X)$ The only class in $A^j(X)$ numerically equivalent to 0 is 0, that is, numerical equivalence and ℓ -adic homological equivalence for algebraic cycles of codimension j on X with rational coefficients.
- $SS^i(X)$ The Galois group $G = \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acts semisimply on $H^i(X)$.
- $S^j(X)$ The map $V^j(X)^G \rightarrow V^j(X)_G$ induced by the identity is bijective.

Claim 1.3.22. *Let X and Y be smooth, projective varieties over \mathbb{F}_q and ℓ coprime with q . If $\zeta(X) = \zeta(Y)$, then, assuming the Tate conjectures, the numerical motives of X and Y with \mathbb{Q}_ℓ -coefficients are isomorphic.*

Proof. Let X and Y be of dimension d . We know by the Lefschetz fixed-point theorem that the sets of eigenvalues of geometric Frobenius acting on $\bigoplus_i (-1)^i H^i(\bar{X}, \mathbb{Q}_\ell)$ and $\bigoplus_i (-1)^i H^i(\bar{Y}, \mathbb{Q}_\ell)$ are equal. By Deligne's theory of weights [17, Théorème 1.6], the eigenvalues of φ^* acting on degree i cohomology have absolute value $q^{\frac{i}{2}}$, so we can conclude that the sets of eigenvalues of φ^* acting on $H^i(X, \mathbb{Q}_\ell)$ and $H^i(Y, \mathbb{Q}_\ell)$ are equal, for each i . By the semisimplicity portion of the above conjectures ($SS^i(X)$ and $SS^i(Y)$), we can induce isomorphisms $H^i(X, \mathbb{Q}_\ell) \cong H^i(\bar{X}, \mathbb{Q}_\ell) \cong H^i(\bar{Y}, \mathbb{Q}_\ell) \cong H^i(Y, \mathbb{Q}_\ell)$ that are invariant under the action of φ^* (and hence that of G).

Maps $H^*(X) \rightarrow H^*(Y)$ correspond to elements of $H^*(X \times Y)$ for H any Weil cohomology theory (cf [24, 1.3]). More, specifically, given an isomorphism $H^i(X) \cong H^i(Y)$, we have by Poincaré duality (A5) that $H^i(X) \cong H^{2d-i}(X)(d)^\vee$, and hence we may identify our isomorphism with an element of $H^{2d-i}(X)(d)^\vee \otimes H^i(Y)$, which we can in turn identify, via Künneth (A4), with an element $\gamma_i \in H^{2d}(X \times Y)(d)$. We identify our set of isomorphisms $\{H^i(X, \mathbb{Q}_\ell) \cong H^i(Y, \mathbb{Q}_\ell)\}_i$ with $\gamma = \bigoplus_i \gamma_i \in H^{2d}(X \times Y)(d)$. The set of isomorphisms can be recovered from γ by pulling elements of $H^*(X, \mathbb{Q}_\ell)$ back to the product $X \times Y$ via the standard projection map, taking the cup product with γ , and pushing the result back down to $H^*(Y, \mathbb{Q}_\ell)$:

$$H^*(X, \mathbb{Q}_\ell) \xrightarrow{p_1^*} H^*(X \times Y, \mathbb{Q}_\ell) \xrightarrow{(-) \cup \gamma} H^*(X \times Y, \mathbb{Q}_\ell) \xrightarrow{p_2^*} H^*(Y, \mathbb{Q}_\ell)$$

By the $I(X \times Y)$ and $T(X \times Y)$ conjectures, we can think of γ as a cycle in $\bigoplus_j \mathbb{Q}_\ell \otimes_{\mathbb{Q}} A^j(X \times Y)$. \square

However, in order to produce a cycle in the Chow group as we are wishing for, more is needed than is implied by the Tate conjectures. We would first want to produce a cycle not just in $\bigoplus_j \mathbb{Q}_\ell \otimes_{\mathbb{Q}} A^j(X \times Y)$, but in $\bigoplus_j A^j(X \times Y)$, for which independence of ℓ would be needed. We would need to show that one can produce a lift of γ in the Chow group of $X \times Y$ that gives an isomorphism between the Chow motives of X and Y .

1.4 The Mukai motive

The Mukai motive has its origins in Mukai's paper [31]; see also [26] for a thorough discussion of realizations of the Mukai motive in the various cohomology theories.

In [36, Proposition 1], it is shown that the Mukai motive, a sum of twisted motives, is invariant under derived equivalence. For any Weil cohomology H over a field k , the cohomological realization of the Mukai motive of a variety X/k is the even Mukai–Hodge structure $\bigoplus_{i=0}^{\dim X} H^{2i}(X/K)(i)$ (see Definition 3.2.1).

To construct a motive with this cohomological realization in the category of Chow motives, it is necessary to have the decomposition of motives into their graded pieces, for which we need to assume the Künneth standard conjecture:

Conjecture 1.4.1 (Künneth standard conjecture, as stated in [24, §2]). Let H be any Weil cohomology theory defined over a field k . For any smooth, projective variety X/k , the Künneth components of the diagonal class Δ , which correspond to the projection operators $\pi^i : H^*(X) \rightarrow H^i(X)$, are algebraic.

Remark 1.4.2. Kleiman gives a discussion of the various implications among the standard conjectures in [24]. The Künneth standard conjecture is also implied by the Tate conjecture; Tate gives a proof in [44, Theorem (3.2)].

Katz and Messing showed that the Künneth standard conjecture holds over finite fields as a consequence of Deligne's proof of the Riemann hypothesis portion of the Weil conjectures [17]:

Theorem 1.4.3 ([23, Theorem 2 (1)]). *Let H be any Weil cohomology theory defined for varieties over a finite field \mathbb{F}_q . Let X be a smooth, projective absolutely irreducible variety over \mathbb{F}_q of dimension n . Then the Künneth components of the diagonal $\Delta \subset X \times X$ are rationally algebraic cycles, independent of the theory H , and are a \mathbb{Q} -linear combination of the graphs of Frobenius and its iterates.*

The projection operators π^i are idempotents, and the Künneth standard conjecture implies that for any variety X and $i \in \mathbb{Z}$, (X, π_i) is an element in the category of effective Chow motives, and it splits π_i (see Definition 1.1.12). The image of (X, π_i) under a Weil cohomology functor is $H^i(X)$.

Definition 1.4.4. Assuming the Künneth standard conjecture, we define the Mukai motive of a smooth, projective variety X inside the category of Chow motives with rational coefficients to be $\bigoplus_{i=-\infty}^{\infty} (X, \pi_{2i}, i)$.

Proposition 1.4.5 ([36, Proposition 1]). *Let X and Y be smooth projective varieties, and let $F : D^b(X) \rightarrow D^b(Y)$ be a fully faithful functor. Then $\tilde{M}(X)_{\mathbb{Q}}$ is a direct summand of the motive $\tilde{M}(Y)_{\mathbb{Q}}$. If, in addition, F is an equivalence, then the motives $\tilde{M}(X)_{\mathbb{Q}}$ and $\tilde{M}(Y)_{\mathbb{Q}}$ are isomorphic.*

Some parts are left out of the following argument; for further details, see the proof of Lemma 3.2.4 showing the invariance of Mukai–Hodge structures under derived equivalence. The argument is very much analogous to this one.

Sketch of proof. Under these hypotheses, by [35, Theorem 3.2.1] F is naturally isomorphic to a Fourier–Mukai functor, call it Φ_P . F has a left adjoint; call its Fourier–Mukai kernel $Q \in D^b(X \times Y)$. We can associate to $P \in D^b(X \times Y)$ the Mukai vector $v(P) = \text{ch}(P) \sqrt{\text{td}(X \times Y)} \in M_{\sim_{\text{rat}}}(k)_{\mathbb{Q}}(X \times Y)$ and similarly a cycle $v(Q)$ to Q (see Definitions 1.2.9 and 1.2.10).

If Φ_P is fully faithful, $\Phi_Q \circ \Phi_P$ is isomorphic to the identity, and the composition $q \circ p$ gives an identity as well, making $\tilde{M}(X)_\mathbb{Q}$ a summand of $\tilde{M}(Y)_\mathbb{Q}$.

If Φ_P is an equivalence, then Φ_Q is fully faithful, and $\Phi_P \circ \Phi_Q$ is naturally isomorphic to the identity functor, so $\tilde{M}(X)_\mathbb{Q}$ is isomorphic to $\tilde{M}(Y)_\mathbb{Q}$. \square

In addition to the even Mukai–Hodge structure, which is the realization of Mukai motive we define above, there is also an odd Mukai–Hodge structure that is preserved under Fourier–Mukai equivalence (Definition 3.2.1).

For any smooth, projective variety X , its odd Mukai–Hodge structure is defined to be $\bigoplus_{i=1}^{d_X} H^{2i-1}(X/K)(i)$. We define an analogous “odd” Mukai motive:

Definition 1.4.6. Assuming the Künneth standard conjecture, we define the Mukai motive of a smooth, projective variety X inside the category of Chow motives with rational coefficients to be $\bigoplus_{i=-\infty}^{\infty} (X, \pi_{2i-1}, i)$.

Claim 1.4.7. *Let X and Y be smooth projective varieties, and let $F : D^b(X) \rightarrow D^b(Y)$ be a fully faithful functor. Then the odd Mukai motive of X is a direct summand of that of Y . If, in addition, F is an equivalence, then the odd Mukai motives of X and Y are isomorphic.*

The proof follows by the same argument as the proof of [36, Proposition 1].

Chapter 2

Fourier–Mukai transforms and their properties

2.1 Fourier–Mukai transforms: definition

Notation 2.1.1. Given a smooth, noetherian variety X defined over a field k , the bounded derived category of the abelian category of coherent sheaves on X is denoted by $D^b(X)$. From here on we will refer to $D^b(X)$ as the derived category of X .

We will not define derived categories here; some introductory texts on the subject are Huybrechts’ book [22] and Căldăraru’s article [11].

Definition 2.1.2. Given two smooth, noetherian varieties X and Y defined over a field k , a *derived equivalence* between them is a k -linear exact equivalence of triangulated categories $D^b(X) \simeq D^b(Y)$ between their derived categories.

Definition 2.1.3. A functor F between derived categories $D^b(X)$ and $D^b(Y)$ is a *Fourier–Mukai transform* if there exists an object $P \in D^b(X \times Y)$, called a *Fourier–Mukai kernel*, such that

$$F \cong \mathbb{R}p_{Y*}(\mathbb{L}p_X^*(-) \otimes^{\mathbb{L}} P) =: \Phi_P, \quad (2.1)$$

where p_X and p_Y are the projections $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$. A Fourier–Mukai transform that is an equivalence of categories is called a *Fourier–Mukai equivalence*. There is notation in (2.1) to indicate that pushforward, pullback, and tensor are all in their derived versions, but such notation will be suppressed in later appearances for compactness of presentation.

Remark 2.1.4. It is not necessarily well-defined to refer to *the* Fourier–Mukai kernel of an exact functor $D^b(X) \rightarrow D^b(Y)$ between derived categories of smooth, noetherian varieties. Kernels of Fourier–Mukai transforms are not always uniquely determined, though the kernels of Fourier–Mukai equivalences are; we will explore this issues further in Section 2.2.

Remark 2.1.5. A note on terminology: elsewhere in the literature, for instance in the reference [6], what we call Fourier–Mukai functors are known as integral functors and the term Fourier–Mukai transform is then reserved for what we call Fourier–Mukai equivalences here.

Proposition 2.1.6 (Mukai [30]; see also Huybrechts [22, Proposition 5.10]). *The composition of two Fourier–Mukai transforms*

$$\Phi_P : D^b(X) \rightarrow D^b(Y) \quad \text{and} \quad \Phi_Q : D^b(Y) \rightarrow D^b(Z)$$

is naturally isomorphic to a Fourier–Mukai transform with kernel $P \boxtimes Q := \pi_{13}(\pi_{12}^* P \otimes \pi_{23}^* Q)$, where π_{ij} is the projection from $X \times X \times Y$ to the product of its i^{th} and j^{th} factors.*

2.2 Existence and uniqueness of Fourier–Mukai transforms and natural transformations between them

In this section we define Fourier–Mukai transforms and give a brief literature survey about results characterizing them and morphisms between them.

Notation 2.2.1. Given triangulated categories T and T' , we denote by $\text{ExFun}(T, T')$ the category whose objects are exact functors from $T \rightarrow T'$ and whose morphisms natural transformations between them.

We frame the discussion of Fourier–Mukai transforms in this section in terms of the following functor mapping Fourier–Mukai kernels to the transforms that they determine:

$$\begin{aligned} \Phi_{(-)} : D^b(X \times Y) &\rightarrow \text{ExFun}(D^b(X), D^b(Y)) \\ P &\mapsto \Phi_P \\ f : P \rightarrow Q &\mapsto \Phi_{(-)}(f) = p_{Y*}(p_X^*(-) \otimes f) : \Phi_P \Rightarrow \Phi_Q \end{aligned} \tag{2.2}$$

To keep the notation minimal, we do not decorate $\Phi_{(-)}$ with an X and Y , but X and Y should be understood as the varieties associated with the domain and codomain of $\Phi_{(-)}$ for the remainder of the section.

The topics are as follows:

- (Q1) What is the essential image of $\Phi_{(-)}$?
- (Q2) When is $\Phi_{(-)}$ essentially injective?
- (Q3) When is $\Phi_{(-)}$ faithful?
- (Q4) When is $\Phi_{(-)}$ full?

Remark 2.2.2. The survey of results by Canonaco and Stellari [13] also uses the functor $\Phi_{(-)}$ as an organizing principle, and for ease of comparison, we have followed their numbering scheme for questions (Q1)–(Q4). Many of the results cited here are in [13], although we mention a few updates from work published after [13]. Also, our focus is only on derived categories of coherent sheaves on varieties; for a discussion that includes DG categories, categories of perfect complexes on varieties, and derived categories of quasicoherent sheaves on varieties, see [13].

(Q1) and (Q2)

The following result shows that all equivalences $F : D^b(X) \rightarrow D^b(Y)$ are in the essential image of $\Phi_{(-)}$, and that the restriction of $\Phi_{(-)}$ to the full subcategory of kernels mapping to equivalences is essentially injective.

Theorem 2.2.3 (Orlov [35, Theorem 3.2.1]). *Let F be an exact functor from $D^b(X)$ to $D^b(Y)$, where X and Y are smooth projective varieties. Suppose that F is fully faithful and has a right (or left) adjoint functor. Then there is an object $\mathcal{E} \in D^b(X \times Y)$ such that F is isomorphic to the functor $\Phi_{\mathcal{E}}$, and the object \mathcal{E} is determined uniquely up to isomorphism.*

Theorem 2.2.4 (Canonaco and Stellari [12, Theorem 1.1], without twists). *Let X and Y be smooth, projective varieties and $F : D^b(X) \rightarrow D^b(Y)$ be an exact functor such that, for \mathcal{F}, \mathcal{G} any coherent sheaves on X ,*

$$\mathrm{Hom}_{D^b(Y)}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0 \quad \text{if } j < 0. \quad (2.3)$$

Then there is an object $P \in D^b(X \times Y)$ such that Φ_P is naturally isomorphic to F , and the object P is determined uniquely up to isomorphism.

Remark 2.2.5. The result [12, Theorem 1.1] is written for varieties twisted by Brauer classes, but here for our purposes, to avoid getting into the extra definitions not used elsewhere in this document, we state a weaker version without the twists.

Claim 2.2.6. *Theorem 2.2.4 implies Theorem 2.2.3, but Theorem 2.2.3 does not imply Theorem 2.2.4.*

Proof. By [13, Proposition 3.5], Theorem 2.2.3 still holds after removing the hypothesis that F have a left or right adjoint. Since X and Y are smooth and projective, all full functors satisfy (2.3).

It is shown in [13, Example 3.10] that for any smooth, projective variety X and $L \in \mathrm{Pic}(X)$, the functor $\Phi_{\Delta_* L} \oplus \Phi_{\Delta^* L} : D^b(X) \rightarrow D^b(X)$ satisfies (2.3) and is not full. \square

However, $\Phi_{(-)}$ is not in general essentially surjective:

Theorem 2.2.7 (Rizzardo and Van den Bergh [38, Theorem 1.4]). *There is an exact functor that is not a Fourier–Mukai transform from the derived category of a smooth quadric in \mathbb{P}^4 to the derived category of \mathbb{P}^4 .*

Remark 2.2.8. However, even when $\Phi_{(-)}$ fails to be essentially surjective, there is still some interesting behavior: for any exact functor between smooth projective varieties over an algebraically closed field there are sheaves indexed by \mathbb{Z} that coincide with the cohomology sheaves of the kernel when the functor is a Fourier–Mukai transform (Rizzardo [37, Theorem 1.1]).

Lemma 2.2.9 ([13, Lemma 4.2]). *If X or Y is \mathbb{P}^1 , then $\Phi_{(-)}$ is essentially injective.*

However, $\Phi_{(-)}$ is not always essentially injective for X and Y that are smooth and projective and even 1-dimensional:

Theorem 2.2.10 ([14, Theorem 1.1]). *For every elliptic curve X over an algebraically closed field, there exist $\mathcal{E}_1, \mathcal{E}_2 \in D^b(X \times X)$ such that $\mathcal{E}_1 \not\cong \mathcal{E}_2$ but $\Phi_{\mathcal{E}_1} \cong \Phi_{\mathcal{E}_2}$.*

(Q3) and (Q4)

$\Phi_{(-)}$ is not necessarily fully faithful, meaning natural transformations between Fourier–Mukai transforms do not necessarily correspond with morphisms between their kernels:

Proposition 2.2.11 (Canonico and Stellari [14, Proposition 2.3]). *If (at least) one of X and Y is 1-dimensional, $\Phi_{(-)}$ is neither full nor faithful.*

However, $\Phi_{(-)}$ does map isomorphisms to natural isomorphisms.

Lemma 2.2.12 (Lieblich and Olsson [26, Lemma 3.4]). *Let $a : P \rightarrow Q$ be a morphism of Fourier–Mukai kernels in $D^b(X \times Y)$. The natural transformation $\Phi_{(-)}(a)$ is an isomorphism if and only if a is an isomorphism.*

Proof. The “if” direction follows immediately from functoriality. In the “only if” direction, by assumption the maps $\Phi_{(-)}(a)_{k(x)} : P_x \rightarrow Q_x$ are isomorphisms. By the Nakayama lemma (for perfect complexes), this implies that a is an isomorphism. \square

2.3 Fourier–Mukai transforms and canonical bundles

In this section, we give a presentation of several results shown in Orlov [35] and Toda [46], and some of their consequences. We start with a proof that derived equivalent smooth, projective varieties have isomorphic canonical rings, and work through several results relating to the supports of kernels of Fourier–Mukai equivalences.

Canonical rings under derived equivalence

Many qualities of varieties that are related to canonical bundles are preserved under derived equivalence. At the heart of these results is the following theorem:

Lemma 2.3.1 (Bondal and Orlov [8, Proposition 1.3]; see also the exposition [22, Lemma 1.30, Theorem 3.12]). *Any derived equivalence $D^b(X) \cong D^b(Y)$ commutes with the functors*

$$S_X = (-) \otimes_{\omega_X} [\dim(X)] : D^b(X) \rightarrow D^b(X) \quad \text{and} \quad S_Y = (-) \otimes_{\omega_Y} [\dim(Y)] : D^b(Y) \rightarrow D^b(Y).$$

Remark 2.3.2. Lemma 2.3.1 is a special case of the result that, given a field k , Serre functors, first defined in [9], commute with k -linear equivalences between k -linear categories that have finite Hom-sets; see [22, Lemma 1.30] for proof of this result.

Serre functors formalize Serre duality, and S_X and S_Y are examples of such functors.

We attribute the following proposition to Orlov as it is a formal consequence of [35, Theorem 3.2.1]:

Proposition 2.3.3 (Orlov). *Derived equivalent smooth, projective varieties X and Y have isomorphic canonical (and anti-canonical) rings.*

The proof here gives the same construction as provided in statements 4.1–4.4 of [46]; see also [22, Proposition 6.1] for a similar proof.

Proof. Recall that the canonical and anti-canonical rings of a variety X are defined to be the direct sums

$$\bigoplus_{\substack{m \in \mathbb{Z} \\ m \geq 0}} H^0(X, \omega_X^{\otimes m}) \quad \text{and} \quad \bigoplus_{\substack{m \in \mathbb{Z} \\ m \leq 0}} H^0(X, \omega_X^{\otimes m}).$$

Let X and Y be derived equivalent smooth, projective varieties. By Theorem 2.2.4, we may assume that their derived equivalence is given by a Fourier–Mukai equivalence Φ_P for some $P \in D^b(X \times Y)$.

To show the (anti-)canonical rings of X and Y are isomorphic, we exhibit for each $m \in \mathbb{Z}$ an isomorphism $\phi_m : H^0(X, \omega_X^{\otimes m}) \rightarrow H^0(Y, \omega_Y^{\otimes m})$, which we define to be the morphism that makes the following diagram commute:

$$\begin{array}{ccc} H^0(X, \omega_X^{\otimes m}) & \overset{\phi_m}{\dashrightarrow} & H^0(Y, \omega_Y^{\otimes m}) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_X(\mathcal{O}_X, \omega_X^{\otimes m}) & & \mathrm{Hom}_Y(\mathcal{O}_Y, \omega_Y^{\otimes m}) \\ \downarrow \Delta_{X*} & & \downarrow \Delta_{Y*} \\ \mathrm{Hom}_{X \times X}(\Delta_{X*} \mathcal{O}_X, \Delta_{Y*} \omega_X^{\otimes m}) & & \mathrm{Hom}_{Y \times Y}(\Delta_{Y*} \mathcal{O}_Y, \Delta_{Y*} \omega_Y^{\otimes m}) \\ \downarrow P \circ (-) & & \downarrow (-) \circ P \\ \mathrm{Hom}_{X \times X}(P, p_X^* \omega_X \otimes P) & \xrightarrow{\tau^m \circ (-)} & \mathrm{Hom}_{Y \times Y}(P, p_Y^* \omega_Y \otimes P) \end{array} \tag{2.4}$$

It then suffices to define each of the maps on the sides and bottom of the diagram (2.4) and show they are all isomorphisms.

The maps $H^0(X, \omega_X^{\otimes m}) \xrightarrow{\sim} \text{Hom}(\mathcal{O}_X, \omega_X^{\otimes m})$ and $H^0(Y, \omega_Y^{\otimes m}) \xrightarrow{\sim} \text{Hom}(\mathcal{O}_Y, \omega_Y^{\otimes m})$ are the standard identifications from the definition of global sections cohomology.

The maps

$$\begin{aligned} \text{Hom}_X(\mathcal{O}_X, \omega_X^{\otimes m}) &\xrightarrow{\Delta_{X*}} \text{Hom}_{X \times X}(\Delta_{X*}\mathcal{O}_X, \Delta_{X*}\omega_X^{\otimes m}), \\ \text{Hom}_Y(\mathcal{O}_Y, \omega_Y^{\otimes m}) &\xrightarrow{\Delta_{Y*}} \text{Hom}_{Y \times Y}(\Delta_{Y*}\mathcal{O}_Y, \Delta_{Y*}\omega_Y^{\otimes m}) \end{aligned}$$

induced by pushing forward by the diagonal maps $\Delta_X : X \rightarrow X \times X$ and $\Delta_Y : Y \rightarrow Y \times Y$ are also isomorphisms.

We define functors $P \circ (-)$ and $(-) \circ P$ as follows:

$$\begin{aligned} P \circ (-) : D^b(X \times X) &\rightarrow D^b(X \times Y) & \text{and} & & (-) \circ P : D^b(X \times Y) &\rightarrow D^b(Y \times Y) \\ \mathcal{E} &\mapsto \mathcal{E} \boxtimes P & & & \mathcal{F} &\mapsto P \boxtimes \mathcal{F} \end{aligned}$$

See Proposition 2.1.6 for the definition of \boxtimes . Given any $\mathcal{E} \in D^b(X \times X)$ and $\mathcal{F} \in D^b(X \times Y)$, $P \circ \mathcal{E}$ is a kernel of $\Phi_P \circ \Phi_{\mathcal{E}}$ and $\mathcal{F} \circ P$ is isomorphic to the kernel of $\Phi_{\mathcal{F}} \circ \Phi_P$; the following diagrams commute up to isomorphism:

$$\begin{array}{ccc} D^b(X \times X) & \xrightarrow{P \circ (-)} & D^b(X \times Y) \\ \downarrow \Phi_{(-)} & & \downarrow \Phi_{(-)} \\ \text{ExFun}(D^b(X), D^b(X)) & \xrightarrow{\Phi_{P \circ (-)}} & \text{ExFun}(D^b(X), D^b(Y)) \end{array}$$

$$\begin{array}{ccc} D^b(Y \times Y) & \xrightarrow{(-) \circ P} & D^b(X \times Y) \\ \downarrow \Phi_{(-)} & & \downarrow \Phi_{(-)} \\ \text{ExFun}(D^b(Y), D^b(Y)) & \xrightarrow{(-) \circ \Phi_P} & \text{ExFun}(D^b(X), D^b(Y)) \end{array}$$

Since Φ_P is a Fourier–Mukai equivalence, it has a quasi-inverse given by a Fourier–Mukai transform $\Phi_Q : D^b(Y) \rightarrow D^b(X)$. And so, $\Phi_P \circ (-)$ and $(-) \circ \Phi_P$ have quasi-inverses $\Phi_Q \circ (-)$ and $(-) \circ \Phi_Q$, and $P \circ (-)$ and $(-) \circ P$ have quasi-inverses $Q \circ (-)$ and $(-) \circ Q$.

Next, observe that the following diagrams 2-commute:

$$\begin{array}{ccc} D^b(X \times X) & \xrightarrow{P \circ (-)} & D^b(X \times Y) \\ \uparrow \Delta_* & \nearrow p_X^*(-) \otimes P & \\ D^b(X) & & \end{array} \quad \begin{array}{ccc} D^b(Y \times Y) & \xrightarrow{(-) \circ P} & D^b(X \times Y) \\ \uparrow \Delta_* & \nearrow p_Y^*(-) \otimes P & \\ D^b(Y) & & \end{array}$$

Hence $P \circ (-)$ and $(-) \circ P$ induce isomorphisms:

$$\begin{aligned} P \circ (-) : \operatorname{Hom}_{D^b(X \times X)}(\Delta_{X*}(-), \Delta_{X*}(-)) &\xrightarrow{\sim} \operatorname{Hom}_{D^b(X \times Y)}(P \circ \Delta_{X*}(-), P \circ \Delta_{X*}(-)) \\ &\xrightarrow{\sim} \operatorname{Hom}_{D^b(X \times Y)}(p_X^*(-) \otimes P, p_Y^*(-) \otimes P), \\ (-) \circ P : \operatorname{Hom}_{D^b(Y \times Y)}(\Delta_{Y*}(-), \Delta_{Y*}(-)) &\xrightarrow{\sim} \operatorname{Hom}_{D^b(X \times Y)}(\Delta_{Y*}(-) \circ P, \Delta_{Y*}(-) \circ P) \\ &\xrightarrow{\sim} \operatorname{Hom}_{D^b(X \times Y)}(p_X^*(-) \otimes P, p_Y^*(-) \otimes P), \end{aligned}$$

giving the isomorphisms

$$\begin{aligned} \operatorname{Hom}_{X \times X}(\Delta_{X*}\mathcal{O}_X, \Delta_{Y*}\omega_X^{\otimes m}) &\xrightarrow{P \circ (-)} \operatorname{Hom}_{D^b(X \times Y)}(P, p_X^*\omega_X^{\otimes m} \otimes P), \\ \operatorname{Hom}_{Y \times Y}(\Delta_{Y*}\mathcal{O}_Y, \Delta_{Y*}\omega_Y^{\otimes m}) &\xrightarrow{(-) \circ P} \operatorname{Hom}_{D^b(X \times Y)}(P, p_Y^*\omega_Y^{\otimes m} \otimes P), \end{aligned}$$

in (2.4).

Finally, we construct the map $\tau^m \circ (-)$ on the bottom of (2.4). Lemma 2.3.1 gives a natural isomorphism between the functors $S_Y \circ \Phi_P \cong \Phi_P \circ S_X$. Thus there is a natural isomorphism $S_Y^m[-md] \circ \Phi_P \cong \Phi_P \circ S_X^m[-md]$ as well, where $S_X[-d]$ where $d = \dim X = \dim Y$ be the Serre functor S_X shifted by d , and $S_X^m[-md]$ is composition of $S_X[-d]$ with itself m times. Since the functors $S_Y^m[-md] \circ \Phi_P$ and $\Phi_P \circ S_X^m[-md]$ are equivalences, by the uniqueness portion of Theorem 2.2.4, implies there is an isomorphism between their kernels: $\tau^m : P \otimes p_X^*\omega_X^{\otimes m} \xrightarrow{\sim} P \otimes p_Y^*\omega_Y^{\otimes m}$. \square

Remark 2.3.4. The proof does not directly use the fact that X and Y are smooth and projective, but instead uses one of the consequences of [35, Theorem 3.2.1]: We could exchange the requirement that X and Y be smooth and projective, in the hypothesis of Proposition 2.3.3 for the requirement that Fourier–Mukai equivalences $D^b(X) \xrightarrow{\sim} D^b(Y)$ have kernels that are unique up to isomorphism.

The following theorem is a direct corollary to Proposition 2.3.3 since any variety with (anti-)ample canonical bundle is isomorphic to Proj of its (anti-)canonical ring.

Corollary 2.3.5 (Weaker version of Bondal and Orlov [8, Theorem 2.5]). *If there is an exact equivalence $D^b(X) \simeq D^b(Y)$ between smooth varieties X and Y , and X is irreducible, projective and has ample or anti-ample canonical bundle, then X is isomorphic to Y .*

Remark 2.3.6. Bondal and Orlov’s result [8, Theorem 2.5] is stronger than Corollary 2.3.5, and is not a corollary of Proposition 2.3.3 as it does not assume that Y is projective.

Canonical bundles and supports

The maps $\phi_m : H^0(X, \omega_X^{\otimes m}) \rightarrow H^0(Y, \omega_Y^{\otimes m})$ constructed in the proof of Proposition 2.3.3 also give information about the supports of images of sheaves under Fourier–Mukai equivalence. In this section we work through proofs of several results in [46].

Notation 2.3.7. For any $\sigma \in H^0(X, \omega_X^{\otimes m})$, we denote its image under ϕ_m by σ^\dagger , following the notation used in [46].

We can identify any element $\sigma \in H^0(X, \omega_X^{\otimes m})$ with a map $\Delta_* \mathcal{O}_X \rightarrow \Delta_* \omega_X^{\otimes m}$. Then, we have the natural transformation Φ_σ (in the notation of (2.2)), which we denote by $\underline{\sigma} : \text{id} \Rightarrow S_X : D^b(X) \rightarrow D^b(X)$, and is given, for any $\mathcal{E} \in D^b(X)$, $\underline{\sigma}_\mathcal{E} : \mathcal{E} \xrightarrow{\otimes \sigma} \mathcal{E} \otimes \omega_X^{\otimes m}$

The following lemma will be instrumental in proving the other results in this section.

Lemma 2.3.8. *Given any $\mathcal{E} \in D^b(X)$, $\text{Supp}(\mathcal{E}) \subseteq (\sigma)_0$ if and only if $\text{Supp}(\Phi_P(\mathcal{E})) \subseteq (\sigma^\dagger)_0$, where $(\sigma)_0$ denotes the zero-locus of σ .*

Proof. The proof of this lemma comes from applying the following two claims together:

Claim 2.3.9. *The support of \mathcal{E} is contained in $(\sigma)_0$ if and only if $\underline{\sigma}_\mathcal{E}^l : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_X^l$ is the zero map for some l .*

Claim 2.3.10. *For any $\mathcal{E} \in D^b(X)$, $\underline{\sigma}_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_X$ is the zero map if and only if $\underline{\sigma}_{\Phi_P(\mathcal{E})}^\dagger : \Phi_P(\mathcal{E}) \rightarrow \Phi_P(\mathcal{E}) \otimes \omega_Y$ is the zero map.* \square

Proof of Claim 2.3.10. Given $\underline{\sigma}$ as above, $\underline{\sigma}^\dagger : \text{id} \Rightarrow S_Y$ is given by composing the transformations in the following diagram:

$$\begin{array}{ccccc}
 D^b(Y) & & & & \\
 & \searrow \Phi_P^{-1} & & & \\
 & & D^b(X) & & \\
 & & \downarrow \text{id} \left(\begin{smallmatrix} \sigma \\ \Rightarrow \end{smallmatrix} \right) S_X & \searrow \Phi_P & \\
 & & D^b(X) & \xrightarrow{\tau} & D^b(Y) \\
 & \swarrow \Phi_P^{-1} & & \swarrow S_Y & \\
 D^b(Y) & & & &
 \end{array}$$

Hence, $\underline{\sigma}_{\Phi_P(\mathcal{E})}^\dagger \cong \Phi_P(\mathcal{E}) \xrightarrow{\Phi_P(\underline{\sigma}_\mathcal{E})} \Phi_P(\mathcal{E} \otimes \omega_X) \xrightarrow{\tau_\mathcal{E}} \Phi_P(\mathcal{E}) \otimes \omega_Y$ (see [46, Lemma 4.5]). By this definition, if $\underline{\sigma}_\mathcal{E}$ is a zero map, then so is $\underline{\sigma}_{\Phi_P(\mathcal{E})}^\dagger$. The converse follows as well since $\underline{\sigma} = \underline{(\sigma^\dagger)}^\dagger$ can be constructed from $\underline{\sigma}^\dagger$. \square

Definition 2.3.11. Given any object a in a derived category $D^b(X)$, we define its (set-theoretic) *support* $\text{Supp } a$ to be $\bigcup \text{Supp } H^i(a)$, the union of the supports of its cohomology sheaves.

Definition 2.3.12. For any scheme X and closed subscheme Z , we define the full subcategory $D_Z^b(X) \subset D^b(X)$ as follows:

$$\text{Ob}(D_Z^b(X)) := \{a \in D^b(X) \mid \text{Supp } a \subset Z\}.$$

Compare parts (a) and (b) of the following corollary to Lemmas 4.6 and 7.4 in [46].

Corollary 2.3.13. (a) *Let $\sigma_i, i \in I$ be a set of elements in $H^0(X, \omega^{\otimes m})$. Define, for each i , $E_i := (\sigma_i)_0$, the zero locus of σ_i , and $E_i^\dagger := (\sigma_i^\dagger)_0$. Then the Fourier–Mukai transform Φ_P maps objects in $D_{\cap_i E_i}^b(X)$ to objects in $D_{\cap_i E_i^\dagger}^b(Y)$.*

(b) *Let Z be the base locus of $\omega_X^{\otimes m}$ and Z^\dagger be the base locus of $\omega_Y^{\otimes m}$. Then Φ_P maps objects in $D_Z^b(X)$ to objects in $D_{Z^\dagger}^b(Y)$.*

(c) *If X and Y are derived equivalent varieties and ω_Y is semiample, then ω_X is semiample.*

Proof. (a) This statement follows directly from Lemma 2.3.8.

(b) Since Z is the intersection of the zero loci of all $\sigma \in H^0(X, \omega^{\otimes m})$, this statement is a special case of (a).

(c) A line bundle is said to be semiample if some tensor power of it is basepoint-free. By part (b), $\omega_X^{\otimes m}$ has an empty base locus if and only if $\omega_Y^{\otimes m}$ does. \square

Proposition 2.3.14. *Let $\Phi_P : D^b(X) \rightarrow D^b(Y)$ be an equivalence between smooth, projective varieties with semiample (anti-)canonical bundles and, hence, shared (anti-)canonical variety S (by Proposition 2.3.3). Then $\text{Supp } |P| \subseteq X \times_S Y$.*

Proof. There is some power of $\omega_X^{\otimes m}$ that is base-point-free and so, by the proof of Corollary 2.3.13(c), $\omega_Y^{\otimes m}$ is also base-point-free. We can use these line bundles to then induce the canonical maps $f : X \rightarrow S$ and $g : Y \rightarrow S$.

For any $s \in S$, $f^{-1}(s)$ can be expressed as an intersection of divisors $\bigcap_i E_i$. Observe that then $g^{-1}(s) = \bigcap_i E_i^\dagger$. By Corollary 2.3.13, if the support of $\mathcal{E} \in D^b(X)$ is contained in $f^{-1}(s)$, then the support of $\Phi_P(\mathcal{E})$ is contained in $g^{-1}(s)$. In particular, for any $x \in f^{-1}(s)$, the complex $P_{\{x\} \times Y} = \Phi_P(k(x))$, where $k(x)$ is the skyscraper sheaf at x , is supported in $g^{-1}(s)$. \square

2.4 Fourier–Mukai transforms and fibers

Adjoints of Fourier–Mukai transforms

Proposition 2.4.1 (Mukai [30]). *Let X and Y be smooth, projective varieties. A Fourier–Mukai transform $\Phi_P : D^b(X) \rightarrow D^b(Y)$ with kernel $P \in D^b(X \times Y)$ has left and right adjoints that are also Fourier–Mukai transforms. The kernels of the left and right adjoints are, respectively*

$$P_L := P^\vee \otimes p_Y^* \omega_Y[\dim(Y)] \quad \text{and} \quad P_R := P^\vee \otimes p_X^* \omega_X[\dim(X)],$$

where P^\vee is defined to be the derived dual $\mathcal{R}\mathcal{H}om(P, \mathcal{O}_{X \times Y})$ (see also [22, Definition 5.7]). The maps p_X and p_Y are the standard projections from $X \times Y$ to X and Y , respectively.

Note that if Φ_P is an equivalence, then P_L and P_R are isomorphic, and $\Phi_{P_L} \cong \Phi_{P_R}$ is the quasi-inverse of Φ_P .

Claim 2.4.2. *Let $G \dashv F : \mathcal{C} \rightarrow \mathcal{D}$ be an adjunction. If the unit $\text{id}_{\mathcal{D}} \Rightarrow F \circ G$ is an isomorphism, then F is an equivalence of categories and G is its quasi-inverse. Likewise, if the counit $G \circ F \Rightarrow \text{id}_{\mathcal{C}}$ is an isomorphism, then F is an equivalence of categories and G is its quasi-inverse.*

See Mac Lane [27, IV] for background information about adjunctions.

Proof. If the unit is an isomorphism, then the natural isomorphism $\text{id}_{\mathcal{D}} \Rightarrow F \circ G$ states that G is a right quasi-inverse to F , and so F is full and essentially surjective and G is faithful and essentially injective.

The adjunction further implies that in each of these cases, both F and G are fully faithful: Given any $X, Y \in \text{ob } \mathcal{D}$, since F is essentially surjective, $\text{Hom}(X, Y) = \text{Hom}(FZ, FW)$ for some $Z, W \in \mathcal{C}$, and so we have the following series of isomorphisms:

$$\begin{aligned} \text{Hom}(X, Y) &= \text{Hom}(FZ, FW) \cong \text{Hom}(FZ, FGFW) \\ &\cong \text{Hom}(GFZ, GFW) = \text{Hom}(GX, GY). \end{aligned} \quad (2.5)$$

The first isomorphism in (2.5) comes from the unit, and the second comes from the definition of the adjunction $G \dashv F$. By (2.5), there is an isomorphism $\text{Hom}(X, Y) \cong \text{Hom}(GX, GY)$, and hence $\text{Hom}(FX, FY) \cong \text{Hom}(FGX, FGY) \cong \text{Hom}(X, Y)$, implying that both F and G are fully faithful.

Since F is fully faithful and essentially injective, it is an equivalence, and G gives its quasi-inverse and so is an equivalence as well.

The proof that if the counit is an isomorphism, then F is an equivalence and G is its quasi-inverse is similar. \square

Kernels on fiber products

Assumption 2.4.3. Let X, Y, S be smooth, projective varieties, let there be a Fourier–Mukai equivalence $\Phi_P : D^b(X) \rightarrow D^b(Y)$, maps $f : X \rightarrow S, g : Y \rightarrow S$, and let $i : X \times_S Y \rightarrow X \times Y$ be an inclusion. Assume that there is a complex $Q \in D^b(X \times_S Y)$ such that $i_*Q \cong P$.

Remark 2.4.4. In the setting of Assumption 2.4.3, note that the functor $\pi_{X*}(\pi_Y^*(-) \otimes Q)$, where $\pi_X : X \times_S Y \rightarrow X$ and $\pi_Y : X \times_S Y \rightarrow Y$ are the projection maps, is isomorphic to the functor Φ_P .

Notation 2.4.5. We denote the functor $\pi_{X*}(\pi_Y^*(-) \otimes Q)$ described in Remark 2.4.4 by Φ_Q ; it will be clear from context whether the kernel being referenced is in $D^b(X \times_S Y)$ or $D^b(X \times Y)$.

Remark 2.4.6. In the setting of Assumption 2.4.3, Φ_{Q_R} and Φ_{Q_L} give left and right adjoints, respectively, to Φ_Q , where

$$Q_L := Q^\vee \otimes \pi_Y^* \omega_Y[\dim(Y)] \quad \text{and} \quad Q_R := Q^\vee \otimes \pi_X^* \omega_X[\dim(X)].$$

Notation 2.4.7. We denote the units of the adjunctions $\Phi_P \dashv \Phi_{P_R}$ and $\Phi_{P_L} \dashv \Phi_P$ by

$$\eta_P : \text{id} \Rightarrow \Phi_{P_R} \circ \Phi_P \quad \text{and} \quad \eta_{P_L} : \text{id} \Rightarrow \Phi_P \circ \Phi_{P_L},$$

and the units of $\Phi_Q \dashv \Phi_{Q_R}$ and $\Phi_{Q_L} \dashv \Phi_Q$ by

$$\eta_Q : \text{id} \Rightarrow \Phi_{Q_R} \circ \Phi_Q \quad \text{and} \quad \eta_{Q_L} : \text{id} \Rightarrow \Phi_Q \circ \Phi_{Q_L}.$$

To distinguish these units from each other, we decorate them with the left adjoints the adjunctions they arise from. As observed in Remark 2.4.4, $\eta_P = \eta_Q$ and $\eta_{P_L} = \eta_{Q_L}$.

Remark 2.4.8. Also, in Theorem 3.14 of [13], it is shown that if an exact functor between the derived categories of smooth projective varieties over an algebraically closed characteristic 0 field is full, then it is faithful as well. However, since we are avoiding making assumptions on our ground field, we will need to show faithfulness in addition to fullness.

Claim 2.4.9. *The natural transformation η_Q is realized by the following natural map on kernels of id and $\Phi_{Q_R} \circ \Phi_Q$ where $\gamma_{X,X}$, $\gamma_{X,Y}$ and $\gamma_{Y,X}$ are projection maps along the first and third, first and second, and second and third factors of $X \times_S Y \times_S X$, respectively, π_X is the projection to the first factor of $X \times_S Y$, and by Δ we mean the natural maps $X \rightarrow X \times_S X$ and $Y \rightarrow Y \times_S Y$:*

$$\bar{\eta}_Q : \Delta_* \mathcal{O}_X \rightarrow \gamma_{X,X*}(\gamma_{X,Y}^* Q \otimes \gamma_{Y,X}^*(Q^\vee \otimes \pi_X^* \omega_X[\dim(X)])).$$

Likewise, η_{P_L} is realized by the following map on kernels

$$\bar{\eta}_{Q_L} : \Delta_* \mathcal{O}_Y \rightarrow \lambda_{Y,Y*}(\lambda_{Y,X}^*(Q^\vee \otimes \pi_Y^* \omega_Y[\dim(Y)]) \otimes \lambda_{X,Y}^* Q)$$

where $\lambda_{Y,Y}$, $\lambda_{Y,X}$ and $\lambda_{X,Y}$ are projection maps along the first and third, first and second, and second and third factors of $Y \times_S X \times_S Y$, respectively, and p_Y is the projection $Y \times_S X \rightarrow Y$.

Notation 2.4.10. Let $s \in S$ be a geometric point, and X_s, Y_s be the fibers of X and Y over s . Given $Q \in D^b(X \times_S Y)$, we denote Q_s by the pullback of Q to X_s .

Let $a : X_s \rightarrow X$ and $b : Y_s \rightarrow Y$ be the inclusions.

The maps $\gamma_{X,X,s}$, $\gamma_{X,Y,s}$ and $\gamma_{Y,X,s}$ denote the projection maps from $X_s \times_{(k(s))} Y_s \times_{(k(s))} X_s$ to its first and third, first and second, and second and third factors. The maps $\lambda_{Y,Y,s}$, $\lambda_{Y,X,s}$ and $\lambda_{X,Y,s}$ are projection maps to the first and third, first and second, and second and third factors of $Y_s \times_{k(s)} X_s \times_{k(s)} Y_s$.

Claim 2.4.11. *Suppose we are in the setting of Assumption 2.4.3. There exists $R \in D^b(Y_s \times_{k(s)} X_s)$ such that $\text{id} \cong \Phi_R \circ \Phi_{Q_s}$.*

Proof. Since Φ_Q is an equivalence, η_{Q_L} is an isomorphism. So by Lemma 2.2.12, $\bar{\eta}_Q$ is an isomorphism as well.

Since $\bar{\eta}_Q$ is an isomorphism, so is its pullback $(a, a)^*\bar{\eta}_Q$:

$$(a, a)^*\bar{\eta}_Q : (a, a)^*\Delta_*\mathcal{O}_X \rightarrow (a, a)^*\gamma_{X,X*}(\gamma_{X,Y}^*P \otimes \gamma_{Y,X}^*(P^\vee \otimes \pi_X^*\omega_X[\dim(X)])).$$

Since the pullbacks and pushforwards in consideration here are derived, the push-pull comparison maps $(a, a)^*\Delta_*(-) \Rightarrow \Delta_*a^*(-)$ induced by the following cartesian diagram are isomorphisms:

$$\begin{array}{ccc} X_s & \xrightarrow{a} & X \\ \Delta \downarrow & & \downarrow \Delta \\ X_s \times_{k(s)} X_s & \xrightarrow{(a,a)} & X \times_S X \end{array} \quad (2.6)$$

So, $(a, a)^*\Delta_*\mathcal{O}_X \cong \Delta_*a^*\mathcal{O}_X = \Delta_*\mathcal{O}_Z$, hence we may write $(a, a)^*\bar{\eta}_Q$ as

$$(a, a)^*\bar{\eta}_Q : \Delta_*\mathcal{O}_{X_s} \rightarrow (a, a)^*\gamma_{X,X*}(\gamma_{X,Y}^*Q \otimes \gamma_{Y,X}^*(Q^\vee \otimes \pi_X^*\omega_X[\dim(X)])).$$

The following cartesian diagram

$$\begin{array}{ccc} X_s \times_{k(s)} Y_s \times_{k(s)} X_s & \xrightarrow{(a,b,a)} & X \times_S Y \times_S X \\ \gamma_{X,X,s} \downarrow & & \downarrow \gamma_{X,X} \\ X_s \times_{k(s)} X_s & \xrightarrow{(a,a)} & X \times_S X \end{array} \quad (2.7)$$

induces an isomorphism on $\gamma_{X,Y}^*Q \otimes \gamma_{Y,X}^*(Q^\vee \otimes \pi_X^*\omega_X[\dim(X)])$:

$$\begin{aligned} (a, a)^*\gamma_{X,X*}(\gamma_{X,Y}^*Q \otimes \gamma_{Y,X}^*(Q^\vee \otimes \pi_X^*\omega_X[\dim(X)])) \\ \rightarrow \gamma_{X,X,s*}(a, b, a)^*(\gamma_{X,Y}^*Q \otimes \gamma_{Y,X}^*(Q^\vee \otimes \pi_X^*\omega_X[\dim(X)])). \end{aligned}$$

Distributing $(a, b, a)^*$ across the tensor product and using the equalities $\gamma_{X,Y} \circ (a, b, a) = (a, b) \circ \gamma_{X,Y,s}$ and $\gamma_{Y,X} \circ (a, b, a) = (b, a) \circ \gamma_{Y,X,s}$, the above isomorphism is equal to the following:

$$\begin{aligned} \gamma_{X,X,s*}(\gamma_{X,Y}^*Q_s \otimes \gamma_{Y,X,s}^*(Q_s^\vee \otimes \pi_X^*\omega_X[\dim(X)])) \\ \rightarrow \gamma_{X,X,s*}(\gamma_{X,Y}^*Q_s \otimes \gamma_{Y,X,s}^*((Q_s)^\vee \otimes (b, a)^*\pi_X^*\omega_X[\dim(X)])). \end{aligned} \quad (2.8)$$

If we set $R := (Q_s)^\vee \otimes (b, a)^*\pi_X^*\omega_X[\dim(X)]$, then the right-hand side of (2.8) is isomorphic to the kernel of $\Phi_R \circ \Phi_{Q_s}$. \square

Claim 2.4.12. *Suppose we are in the setting of Assumption 2.4.3. There exists $T \in D^b(X_s \times Y_s)$ such that $\text{id} \cong \Phi_{Q_s} \circ \Phi_T$.*

Proof. The proof of this claim is immediate since it is the same as Claim 2.4.11, with X exchanged with Y ; the kernel T of the right quasi-inverse of Φ_{Q_s} is $T := (Q_s)^\vee \otimes \pi_Y^*\omega_Y[\dim(Y)]$. \square

Proposition 2.4.13. *Let X and Y be smooth projective varieties with semiample (anti-)canonical bundles and let $\Phi_P : D^b(X) \rightarrow D^b(Y)$ a derived equivalence. Let S be their shared (anti-)canonical variety and $s \in S$ a geometric point. Let $i : X \times_S Y \rightarrow X \times Y$ be the inclusion map. If there is $Q \in D^b(X \times_S Y)$ such that $i_* Q \cong P$ and X_s and Y_s are smooth, then X_s and Y_s are derived equivalent as varieties over $k(s)$.*

Proof. The hypothesis of the proposition satisfies Assumption 2.4.3, where the roles of the map f and g are filled by the canonical maps $X \rightarrow S$ and $Y \rightarrow S$. By Claims 2.4.11 and 2.4.12, Φ_{Q_s} has both a left and right inverse, making Φ_{Q_s} fully faithful and essentially surjective, hence an equivalence of categories. \square

Chapter 3

Isomorphism of Mukai–Hodge structures, and its consequences

In this section, so that we may pursue a method for comparing the zeta functions of derived equivalent varieties using the Lefschetz fixed-point theorem, we discuss the actions of Fourier–Mukai transforms on Weil cohomologies. These maps on cohomology do not in general induce isomorphisms on cohomology groups in each degree, but we do have two Weil-cohomological invariants of derived equivalent varieties: their even and odd Mukai–Hodge structures. These structures are the cohomological realization of the isomorphism of Mukai motives (see Section 1.4). These structures were used in [26] to prove that derived equivalent K3 surfaces over finite fields have equal zeta functions, a result which is extended here.

In the remainder of the chapter, we use these invariants, along with a symmetry among cohomology groups, to show that derived equivalence between some types of varieties over finite fields – including surfaces and Calabi–Yau 3-folds – implies equality of zeta functions.

3.1 Fourier–Mukai transforms acting on cohomology

One of the reasons it is so interesting to know whether a functor between derived categories of smooth, projective varieties over a field k is a Fourier–Mukai transform is that one may use the Fourier–Mukai kernel to induce a map between the K -groups, Chow groups and, for many types of cohomology, the cohomology groups of the varieties concerned. And, when the Fourier–Mukai transform is an equivalence, the induced maps are isomorphisms. This presentation will focus on the maps induced on Chow groups and Weil cohomologies (see Chapter 1 for an introduction to these theories) since the others are not necessary for our purposes; the interested reader may wish to consult [22, Section 5.2] for a discussion of the maps on K -groups and the cohomology of the constant sheaf \mathbb{Q} on complex manifolds.

Definition 3.1.1. Given a smooth, proper variety X over a field k , for any $\mathcal{E} \in D^b(X)$, we

define the *Mukai vector* of \mathcal{E} to be

$$v(\mathcal{E}) := \text{ch}(\mathcal{E}) \cdot \sqrt{\text{td}(X)},$$

which is an element of the Chow group $\text{CH}(X)$. We also use the notation $v(\mathcal{E})$ and the term “Mukai vector” to refer to the image of the Mukai vector in any Weil cohomology group H defined over k via the cycle map ((D5) in the definition of Weil cohomology theories in Section 1.2); whether the Mukai vector being referred to is in the Chow group or a Weil cohomology theory will be clear from context.

Definition 3.1.2. Given smooth, proper varieties X and Y over a field k of dimensions d_X and d_Y , respectively, and a Fourier–Mukai transform $\Phi_P : D^b(X) \rightarrow D^b(Y)$ with kernel $P \in D^b(X \times Y)$, we induce the following map on Chow groups:

$$\Psi_P^{\text{CH}} = p_{Y*}(v(P) \cup p_X^*(-)) : \text{CH}(X) \rightarrow \text{CH}(Y).$$

For any Weil cohomology theory H defined over k , we induce the following map on cohomology:

$$\Psi_P := p_{Y*}(\text{cl}(v(P)) \cup p_X^*(-)). \quad (3.1)$$

In particular, the map induced on the i^{th} cohomology group by the portion of Ψ_P from $v^j(P)$, the the degree $2j$ part of $v(P)$, where $0 \leq i \leq 2d_X$ and $0 \leq j \leq d_X + d_Y$, is

$$\Psi_P^{i,j} : H^i(X/K) \xrightarrow{p_X^*} H^i(X \times Y/K) \xrightarrow{\cup v^j(P)} H^{i+2j}(X \times Y/K)(j) \xrightarrow{p_{Y*}} H^{i+2(j-d_X)}(Y/K)(j-d_X).$$

The Tate twists present in this map are defined and discussed in Section 1.2.

We note that the maps Ψ_P^{CH} and Ψ_P are of much the same form as the Fourier–Mukai transform, where tensoring with the kernel has been replaced by taking the cup product with the Mukai vector of the kernel.

Note however that in addition to introducing Tate twists as shown in Definition 3.1.2, the map Ψ_P does not in general preserve cohomological degrees. For example, given an abelian variety A and its dual \hat{A} , the Fourier–Mukai transform $D^b(A) \rightarrow D^b(\hat{A})$ with kernel the Poincaré bundle P_A is an equivalence (see Mukai [30, Theorem 2.2]), and the map Ψ_{P_A} it induces on cohomology sends $H^i(A)$ isomorphically to $H^{2d-i}(\hat{A})$. Huybrechts gives a proof of this in [22, Lemma 9.23] in the process of showing that the map induced on cohomology by Ψ_{P_A} differs from the maps on cohomology by Poincaré duality by a sign. Huybrechts’ proof refers to varieties over \mathbb{C} and cohomology of the constant sheaf \mathbb{Q} of the varieties’ associated complex manifolds, but the proof hinges on the fact that for each n , the degree $2n$ portion of the Mukai vector of P_A in $H^{2n}(A \times \hat{A}, \mathbb{Q})$ is supported in $H^{2n}(A, \mathbb{Q}) \otimes H^{2n}(\hat{A}, \mathbb{Q})$, a fact which holds in each Weil cohomology theory as well.

So, we need to be careful about the domain and codomain of Ψ_P , which we discuss further in the next subsection.

Uniqueness of Mukai vectors

As shown in Section 2.2 in the discussion of question (Q2), an exact functor $F : D^b(X) \rightarrow D^b(Y)$ between the derived categories of smooth, projective varieties X and Y is in some cases isomorphic to a Fourier–Mukai functor with a uniquely determined kernel, for instance when F is an equivalence (Theorem 2.2.3), or F may be isomorphic to Fourier–Mukai transforms with non-isomorphic kernels. In the following result we show that in either case it is well-defined to refer to the Mukai vector of F .

Proposition 3.1.3 (Corollary of Canonaco and Stellari [14, Theorem 1.2]). *Let X and Y be smooth, projective varieties over a field k and $F : D^b(X) \rightarrow D^b(Y)$ an exact functor isomorphic to a Fourier–Mukai functor Φ_P . The cohomology sheaves of P are uniquely determined by F , and consequently, the class of P in the Grothendieck group $K(X \times Y)$, $\bigoplus_i (-1)^i [\mathcal{H}^i(P)]$, is uniquely determined by F , and the Mukai vector $v(P)$ is uniquely determined by F .*

See [14, Theorem 1.2] for a proof and also [13, Section 4] for further discussion.

3.2 Isomorphism of Mukai–Hodge structures...

Let H be an arbitrary Weil cohomology with coefficients in a characteristic 0 field K , as defined in Section 1.2. Recall we denote the i^{th} cohomology group of X with a Tate twist of n as $H^i(X)(n)$.

Definition 3.2.1. We take the *even and odd Mukai–Hodge structures* of a dimension d_X smooth, projective variety X to be the sums of cohomology groups given by:

$$\begin{aligned}\tilde{H}^{\text{even}}(X/K) &:= \bigoplus_{i=0}^{d_X} H^{2i}(X/K)(i), \\ \tilde{H}^{\text{odd}}(X/K) &:= \bigoplus_{i=1}^{d_X} H^{2i-1}(X/K)(i).\end{aligned}$$

3.2.2 Weights of Mukai–Hodge structures. Each of the Mukai–Hodge structures is of pure weight in the sense of Deligne’s theory of weights [17]. In the case where X is defined over a finite field and H is theory of étale cohomology with \mathbb{Q}_ℓ coefficients, being of pure weight means that all the eigenvalues of the action of geometric Frobenius have the same absolute value. To verify that the even and odd Mukai–Hodge structures are of pure weight, it suffices to show that the presence of a Tate twist by l multiplies the eigenvalues of a cohomology group by $\frac{1}{q^l}$.

To see this, recall (see Chapter 1.1) that Tate twists can be defined relative to the cohomology of the projective line: $K(-1) \cong H^2(\mathbb{P}^1)$. Taking ℓ -adic étale cohomology of the Kummer sequence yields a surjective map $\text{Pic}(\mathbb{P}^1) \rightarrow H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$. The image of the line bundle $\mathcal{O}(1)$, which generates $\text{Pic}(\mathbb{P}^1)$, then also generates $H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$. The image of $\mathcal{O}(1)$

under the action of φ^* is $\mathcal{O}(q)$ (where q is the cardinality of the field we're working over). Thus the eigenvalue of the action of Frobenius on $K(-1)$ is q , so adding a Tate twist of 1 to a vector space multiplies its eigenvalues under the action of Frobenius by $\frac{1}{q}$.

Definition 3.2.3. Let $\Phi_P : D^b(X) \rightarrow D^b(Y)$ be a Fourier–Mukai transform between smooth, projective varieties X and Y . We induce maps Ψ_P^{even} and Ψ_P^{odd} on their even and odd Mukai–Hodge structures, respectively, as follows:

$$\Psi_P^{\text{even}} := \bigoplus_{i=0}^{d_X} \sum_{j=0}^{d_X+d_Y} \Psi_P^{2i,j}(i-d) : \tilde{H}^{\text{even}}(X/K) \rightarrow \tilde{H}^{\text{even}}(Y/K), \quad (3.2)$$

$$\Psi_P^{\text{odd}} := \bigoplus_{i=1}^{d_X} \sum_{j=0}^{d_X+d_Y} \Psi_P^{2i-1,j}(i-d) : \tilde{H}^{\text{odd}}(X/K) \rightarrow \tilde{H}^{\text{odd}}(Y/K), \quad (3.3)$$

We use the notation $\Psi_P^{i,j}$ for the map on the i^{th} cohomology group induced by the degree $2j$ part of $v(P)$ as in Definition 3.1.2 and denote the map on the i^{th} cohomology group twisted by l induced by $\Psi_P^{i,j}$ by $\Psi_P^{i,j}(l) : H^i(X/K)(l) \rightarrow H^{i+2j-2d_X}(Y/K)(l+j-d_X)$.

Note that Ψ_P^{even} and Ψ_P^{odd} are well-defined since their domain and codomain have been chosen appropriately. We can also think of them as being given by applying the map Ψ_P (3.1) to the even and odd Mukai–structures of X , respectively.

Lemma 3.2.4. *Given smooth, projective varieties X and Y and a Fourier–Mukai equivalence $\Phi_P : D^b(X) \rightarrow D^b(Y)$, the maps Ψ_P^{even} and Ψ_P^{odd} are isomorphisms.*

Proof. By Proposition 2.4.1, there is a $P' \in D^b(X \times Y)$ such that $\Phi_{P'}$ is quasi-inverse to Φ_P . Furthermore, $\Phi_{P'} \circ \Phi_P \cong \text{id}_{D^b(Y)} \cong \Phi_{\mathcal{O}_{\Delta_Y}}$ and $\Phi_P \circ \Phi_{P'} \cong \text{id}_{D^b(X)} \cong \Phi_{\mathcal{O}_{\Delta_X}}$, where $\mathcal{O}_{\Delta_X} \in D^b(X \times X)$ and $\mathcal{O}_{\Delta_Y} \in D^b(Y \times Y)$ are the pushforwards of the structure sheaves of the diagonals.

Hence, in order to show that Ψ_P^{even} and Ψ_P^{odd} are isomorphisms, it suffices to prove the following two statements.

- (1) For $Q \in D^b(X \times Y)$, $R \in D^b(Y \times Z)$, $S \in D^b(X \times Z)$ such that Φ_R , Φ_Q and Φ_S are equivalences and $\Phi_R \circ \Phi_Q \cong \Phi_S$, we have $\Psi_R^{\text{even}} \circ \Psi_Q^{\text{even}} \cong \Psi_S^{\text{even}}$ and $\Psi_R^{\text{odd}} \circ \Psi_Q^{\text{odd}} \cong \Psi_S^{\text{odd}}$.
- (2) $\Psi_{\mathcal{O}_{\Delta}}^{\text{even}}$ and $\Psi_{\mathcal{O}_{\Delta}}^{\text{odd}}$ act identically.

(1) By Mukai [30, Proposition 1.3], $\Phi_R \circ \Phi_Q \cong \Phi_{S'}$ for $S' = \pi_{XZ*}(\pi_{XY*}R \otimes \pi_{YZ*}Q)$, where π_{XY} , π_{YZ} , and π_{XZ} are the projection maps from $X \times Y \times Z$ to $X \times Y$, $Y \times Z$, and $X \times Z$. Since $\Phi_{S'}$ and Φ_S are naturally isomorphic and X and Y are smooth and projective, $S \cong S'$ (see Theorem 2.2.3), and so by Proposition 3.1.3 $\Psi_S = \Psi_{S'}$.

Mukai's argument shows directly that $\Phi_R \circ \Phi_Q$ and Φ_S have isomorphic kernels using the projection formula and the flat base change theorem. The same arguments can be applied inside the Chow ring to show that $\Psi_R^{\text{CH}} \circ \Psi_Q^{\text{CH}} = \Psi_S^{\text{CH}}$, and they still apply after we descend

to a Weil cohomology theory of our choice by taking cycle classes. We note that Huybrechts’ proof of a result analogous to [30, Proposition 1.3] for realizations of the Fourier–Mukai functor acting on the cohomology $H^*(X, \mathbb{Q})$ of the constant sheaf \mathbb{Q} on complex manifolds X [22, Lemma 5.32] rests on this same argument – showing Mukai’s proof still works after descending to this particular cohomology theory.

(2) As shown in the proof of [22, Proposition 5.33], as a direct consequence of the Grothendieck–Riemann–Roch formula, for any smooth, projective variety X ,

$$\Psi_{\mathcal{O}_\Delta} = p_{X*}(p_Y^*(-) \cup v(\mathcal{O}_\Delta))$$

acts identically on cohomology groups. Since $\Psi_{\mathcal{O}_\Delta}^{\text{even}}$ and $\Psi_{\mathcal{O}_\Delta}^{\text{odd}}$ are given by the action of $\Psi_{\mathcal{O}_\Delta}$ on $\tilde{H}^{\text{even}}(X/K)$ and $\tilde{H}^{\text{odd}}(Y/K)$, respectively, they each act identically. \square

Remark 3.2.5. 1. The proof of Lemma 3.2.4 shows that when Φ_P is an equivalence, Ψ_P is invertible. Although this proof only addresses the case of Weil cohomologies, the same argument works for Chow groups, to show that Ψ_P^{CH} is invertible as well.

2. Statements (1) and (2) in the proof of Lemma 3.2.4 are establishing a notion of functoriality, but we refrain from putting the proof in that terminology to avoid the unnecessary work of establishing which categories the functors are mapping between.
3. The choice of weight for the even and odd Mukai–Hodge structures was an arbitrary one. We could alter either structure by twisting it by the same amount in each dimension (note the structures would remain pure weight), and the maps induced by Ψ_P would still be well defined and, if Φ_P is an equivalence, invertible.
4. Just as in Proposition 1.4.5, if Φ_P is fully faithful and not necessarily an equivalence, we can use the ideas in the proof of Lemma 3.2.4 to show that the even or odd Mukai–Hodge structure of X is a summand of the even or odd, respectively, Mukai–Hodge structure of Y ; we simply use the fact that in this case Φ_P has a left adjoint Φ_Q such that $\Phi_Q \circ \Phi_P \cong \text{id}$ and so $\Psi_Q^{\text{even}} \circ \Psi_P^{\text{even}} \cong \text{id}$ and $\Psi_Q^{\text{odd}} \circ \Psi_P^{\text{odd}} \cong \text{id}$.

3.3 ... and its consequences

In this section we use the information about cohomology of derived equivalent smooth, projective varieties to compare their zeta functions using the Lefschetz fixed point theorem.

Although our strategy will show that derived equivalent curves over finite fields have equal zeta functions, this is a case that is already well understood, and we give a discussion of it before moving onward.

3.3.1 Derived equivalent curves. In any characteristic, curves of genus 0 (the projective line) have anti-ample canonical bundle and curves of genus at least 2 have ample canonical bundle, implying that in those cases, derived equivalent curves are isomorphic (see Theorem 2.3.5). The characterization of derived equivalent genus 1 was completed in [4], which also

contains a summary of the previous work: it was already known (see [10] for a proof and [22] for further discussion) that over a separably closed field, derived equivalent curves must be isomorphic. It is shown, however, in [4, Corollary 1.2] that derived equivalent nonisomorphic genus 1 curves exist, and all such examples arise between homogeneous spaces of the same elliptic curve. That is, all derived equivalent non-isomorphic curves do not have rational points over their field of definition and so are specifically genus 1 curves, but not elliptic curves, though they are elliptic curves after base change. However, as shown in [4, Example 2.8], such an example cannot arise over a finite field \mathbb{F}_q since Lang’s theorem implies that any genus 1 curve must have a rational point.

3.3.2 Lefschetz fixed-point and comparing zeta functions. By the Lefschetz fixed-point formula for Weil cohomologies (see Section 1.3), to prove smooth, projective varieties X and Y over a finite field have equal zeta functions it suffices to show that, for some Weil cohomology H , the traces of the Frobenius map φ acting on $H^i(X/K)$ and $H^i(Y/K)$ are the same for all i .

In the case where H is ℓ -adic étale cohomology with \mathbb{Q}_ℓ -coefficients, this condition can be shown to be necessary as well:

Claim 3.3.3. *Let X and Y be smooth, projective varieties over a finite field \mathbb{F}_q . $\zeta(X) = \zeta(Y)$ if and only if $\text{Tr}(\varphi^*|H^i(X, \mathbb{Q}_\ell)) = \text{Tr}(\varphi^*|H^i(Y, \mathbb{Q}_\ell))$ for all i .*

Proof. The equality of traces suffices to show the equality of zeta functions by the Lefschetz fixed-point theorem (Theorem 1.3.1).

Given $\zeta(X) = \zeta(Y)$, by the Lefschetz fixed-point theorem, we have the equalities

$$\sum_i (-1)^i \text{Tr}(\varphi^{m*}|H^i(X, \mathbb{Q}_\ell)) = \sum_i (-1)^i \text{Tr}(\varphi^{m*}|H^i(Y, \mathbb{Q}_\ell))$$

for all m . These equalities show that the sum of the m^{th} powers of the eigenvalues of $\bigoplus_i (-1)^i H^i(X, \mathbb{Q}_\ell)$ and the sum of the m^{th} powers of the eigenvalues of $\bigoplus_i (-1)^i H^i(Y, \mathbb{Q}_\ell)$ are equal for any m . Hence, the characteristic polynomials of φ acting on $\bigoplus_i (-1)^i H^i(X, \mathbb{Q}_\ell)$ and $\bigoplus_i (-1)^i H^i(Y, \mathbb{Q}_\ell)$ are equal, implying the sets of eigenvalues of the action of φ on $\bigoplus_i (-1)^i H^i(X, \mathbb{Q}_\ell)$ and $\bigoplus_i (-1)^i H^i(Y, \mathbb{Q}_\ell)$ are equal. Deligne proved the Riemann hypothesis portion of the Weil Conjectures in [17, Théorème 1.6], which states that the eigenvalues of the action of Frobenius on the i^{th} cohomology groups of smooth, projective varieties have absolute value $q^{i/2}$. Thus, if $\zeta(X) = \zeta(Y)$, the eigenvalues of φ acting on $H^i(X, \mathbb{Q}_\ell)$ and $H^i(Y, \mathbb{Q}_\ell)$ are equal, for each i . \square

Comparing zeta functions

Given smooth, projective varieties X and Y , and a Fourier–Mukai equivalence Φ_P , Lemma 3.2.4 shows that the maps Ψ_P^{even} and Ψ_P^{odd} are isomorphisms between the even and odd Mukai–Hodge structures of X and Y . In the case when X and Y are defined over a finite field \mathbb{F}_q , in order to use Lemma 3.2.4 to compare the zeta functions, we show that Ψ_P^{even} and Ψ_P^{odd} are compatible with the action of Frobenius in the following lemma.

Lemma 3.3.4. *Given smooth, projective varieties X and Y over a finite field \mathbb{F}_q and a Fourier–Mukai equivalence $\Phi_P : D^b(X) \rightarrow D^b(Y)$, the map $\Psi_P = p_{Y*}(\text{cl}(v(P)) \cup p_X^*(-))$, and hence the maps Ψ_P^{even} and Ψ_P^{odd} , are compatible with the action of geometric Frobenius: $\Psi_P \circ \varphi_X^* = \varphi_Y^* \circ \Psi_P$.*

Proof. We prove the lemma by checking separately that the action of Frobenius commutes with the maps $p_X^* : H^i(X)(l) \rightarrow H^i(X \times Y)(l)$, $\text{cl}(v(P)) \cup (-) : H^i(X \times Y)(l) \rightarrow H^i(X \times Y)(l)$ and $p_{Y*} : H^i(X \times Y)(l) \rightarrow H^{i-2\dim(X)}(l - \dim(X))$ for any integers i, l .

By the functoriality of geometric Frobenius, the following diagram commutes for any i, l :

$$\begin{array}{ccc} H^i(X)(l) & \xrightarrow{p_X^*} & H^i(X \times Y)(l) \\ \varphi_X^* \downarrow & & \downarrow \varphi_{X \times Y}^* \\ H^i(X)(l) & \xrightarrow{p_X^*} & H^i(X \times Y)(l) \end{array}$$

Therefore, the action of Frobenius commutes with the pullback p_X^* .

To show the compatibility of cup product with the Mukai vector with pullback by the geometric Frobenius map, we check that for any i, l the following diagram commutes:

$$\begin{array}{ccc} H^i(X \times Y)(l) & \xrightarrow{(-) \cup v^j(P)} & H^i(X \times Y)(l) \\ \varphi_{X \times Y}^* \downarrow & & \downarrow \varphi_{X \times Y}^* \\ H^i(X \times Y)(l) & \xrightarrow{(-) \cup v^j(P)} & H^i(X \times Y)(l) \end{array}$$

The composition of the top and right maps in the diagram map any element $w \in H^i(X \times Y)(l)$ to $\varphi^*(w \cup v^j(P)) = \varphi^*w \cup \varphi^*v^j(P)$, and the composition of the left and bottom maps in the diagram map w to $\varphi^*w \cup v^j(P)$. We note that $v^j(P)$ is the image of a cycle class in $H^{2j}(X, \mathbb{Q}_\ell(j))$ (recall (D5) in the definition of a Weil cohomology in Section 1.2). The map φ^* acts on cycles of codimension r by multiplying them by a factor of q^r (for instance, a given a line bundle L , $\varphi^*L = L^{\otimes n}$). In $H^{2j}(X, \mathbb{Q}_\ell(j))$, any eigenvectors of φ have absolute value 1: the Tate twist of j multiplies the images of the cycles inside the cohomology groups by $\frac{1}{q^j}$. Hence, $\varphi^*v^j(P) = v^j(P)$, and the above diagram commutes.

Finally we check the compatibility of φ^* with p_{Y*} , recalling its action on Weil cohomology groups (see Section 1.2) is defined as follows:

$$\begin{aligned} H^j(X \times Y) &\xrightarrow{(A5)} H^{2(d_X+d_Y)-j}(X)(d_X+d_Y)^\vee \xrightarrow{(p_Y^*)^\vee} H^{2(d_X+d_Y)-j}(Y)(d_X+d_Y)^\vee \\ &\xrightarrow{(A5)} H^{-2d_X+j}(Y)(-d_X) \end{aligned}$$

where d_X and d_Y are the dimensions of X and Y . Since the middle map is simply a dualized pullback, it only remains to show that φ^* is compatible with the maps induced by (A5) in our list of Weil cohomology axioms, Poincaré duality: it suffices to show that for any smooth,

projective variety V and any integer j , the following diagram commutes:

$$\begin{array}{ccc} H^j(V) & \xrightarrow{(A5)} & H^{2\dim V-j}(V)(\dim V)^\vee \\ \downarrow \varphi^* & & \uparrow \varphi^* \\ H^j(V) & \xrightarrow{(A5)} & H^{2\dim V-j}(V)(\dim V)^\vee \end{array}$$

Let u be an arbitrary element of $H^j(V)$. Following the top and bottom and sides of the above diagram, it gets sent to the functionals $u \cup (-)$ and $\varphi^*u \cup \varphi^*(-)$. To see that these are the same, consider the action of φ^* on the Poincaré duality map

$$H^j(V) \times H^{2\dim V-j}(V)(\dim V) \xrightarrow{\cup} H^{2\dim V}(V)(\dim V).$$

Since the action of φ^* on $H^{2\dim V}(V)(\dim V)$ is trivial, for any $u \in H^j(V)$ and $w \in H^{2\dim V-j}(V)(\dim V)$, $\varphi^*u \cup \varphi^*w = u \cup w$. If we are considering the map (A5) acting on a cohomology group with Tate twists present, it is still compatible with Frobenius; we can adjust the argument by altering the Poincaré duality mapping as necessary, eg, $H^j(V)(l) \times H^{2\dim V-j}(V)(\dim V - l) \xrightarrow{\cup} H^{2\dim V}(V)(\dim V)$. \square

3.3.5. It would be tempting to look at the isomorphism in (3.2) and hope that the eigenvalues of Frobenius acting on each degree of cohomology could be matched up by using their weights to distinguish them: in Deligne’s [17, Théorème 1.6], it is shown that the eigenvalues of the action of φ^* on $H^i(X, \mathbb{Q}_\ell)$ have absolute value $q^{i/2}$. However, as mentioned in the previous section, the even and odd Mukai–Hodge structures are of pure weight: the twists present in the equations in (3.2) affect the eigenvalues of the action of Frobenius on the cohomology groups in such a way that all the eigenvalues have the same absolute values.

What (3.2) does tell us is that if varieties X and Y are Fourier–Mukai equivalent, the sets of eigenvalues of Frobenius acting on all their even (respectively, odd) cohomology groups, multiplied by factors of $\frac{1}{q}$ until they all have the same absolute value, are equal. This observation is simply a consequence of the fact that the characteristic polynomials of Frobenius acting on their Mukai–Hodge structures are equal.

This information is, on its own, enough to decide that the zeta functions of some derived equivalent varieties are equal, when there are few enough cohomology groups to compare, especially once we use the fact that the traces of Frobenius acting on top and bottom cohomology are always the same (1 and q^d for a dimension d variety, respectively). For instance, we can now prove that derived equivalent K3 surfaces have equal zeta functions, since they have vanishing odd cohomology (this is the method used to prove [26, Theorem 4.1]). This information also gives an alternate argument to that discussed in 3.3.1 that derived equivalent curves over a finite field \mathbb{F}_q have equal zeta functions.

Theorem 3.3.6. *Let X and Y be smooth, projective varieties of dimension d over a finite field \mathbb{F} such that $D^b(X)$ is equivalent to $D^b(Y)$. Let φ be the geometric Frobenius endomorphism. If we have that $\text{Tr}(\varphi^*|H^i(X, \mathbb{Q}_\ell)) = \text{Tr}(\varphi^*|H^i(Y, \mathbb{Q}_\ell))$ for $\lfloor \frac{d}{2} \rfloor - 1$ even values and $\lfloor \frac{d}{2} \rfloor - 1$ odd values of $1 \leq i \leq d$, then $\zeta(X) = \zeta(Y)$.*

Proof. The isomorphisms $\tilde{H}(X, \mathbb{Q}_\ell)^{\text{even}} \cong \tilde{H}(Y, \mathbb{Q}_\ell)^{\text{even}}$ and $\tilde{H}(X, \mathbb{Q}_\ell)^{\text{odd}} \cong \tilde{H}(Y, \mathbb{Q}_\ell)^{\text{odd}}$ of Lemma 3.2.4 imply that

$$\sum_{i=0}^d \text{Tr}(\varphi^* | H^{2i}(X, \mathbb{Q}_\ell)(i)) = \sum_{i=0}^d \text{Tr}(\varphi^* | H^{2i}(Y, \mathbb{Q}_\ell)(i)), \quad (3.4)$$

$$\sum_{i=1}^d \text{Tr}(\varphi^* | H^{2i-1}(X, \mathbb{Q}_\ell)(i)) = \sum_{i=1}^d \text{Tr}(\varphi^* | H^{2i-1}(Y, \mathbb{Q}_\ell)(i)), \quad (3.5)$$

where d is the dimension of X (and of Y ; derived equivalent varieties have equal dimension).

Recall from 3.2.2 that the presence of Tate twists affects the eigenvalues of the action of Frobenius as follows:

$$\text{Tr}(\varphi^* | H^i(X, \mathbb{Q}_\ell)(l)) = \frac{1}{q^l} \text{Tr}(\varphi^* | H^i(X, \mathbb{Q}_\ell)).$$

We can make further progress in comparing zeta functions by using the symmetry among cohomology groups given to us by Hard Lefschetz (Lemma 3.3.7 can also be achieved using Poincaré duality instead of Hard Lefschetz). We will now switch, for the remainder of this proof, from working over an arbitrary Weil cohomology theory to working with ℓ -adic étale cohomology since we will need some facts that have been proven for this cohomology theory in particular.

Lemma 3.3.7. *Let V be a smooth, projective variety of dimension d over the field \mathbb{F}_q and H be étale cohomology with \mathbb{Q}_ℓ -coefficients for $(\ell, q) = 1$. If the eigenvalues (with multiplicity) of φ^* acting on $H^i(V/\mathbb{Q}_\ell)$, $0 \leq i < \frac{d}{2}$, are $\{\alpha_1, \dots, \alpha_n\}$, then the eigenvalues of φ^* acting on $H^{2d-i}(V/\mathbb{Q}_\ell)$ are $\{q^{d-i}\alpha_1, \dots, q^{d-i}\alpha_n\}$.*

Proof of lemma. The Hard Lefschetz Theorem for ℓ -adic étale cohomology (Deligne [16, Théorème 4.1.1]) states that the map $L^{d-i} : H^i(V/\mathbb{Q}_\ell)(i-d) \xrightarrow{\sim} H^{2d-i}(V/\mathbb{Q}_\ell)$ is an isomorphism, where L^{d-i} is the $(d-i)^{\text{th}}$ iteration of the Lefschetz operator L , which is given by intersecting with the hyperplane class. Since L commutes with the action of the Frobenius map on cohomology, the Hard Lefschetz Theorem shows that if the eigenvalues of the action of φ^* on $H^i(V/\mathbb{Q}_\ell)$ are $\{\alpha_1, \dots, \alpha_n\}$, then the eigenvalues of the action of φ^* on $H^{2d-i}(V/\mathbb{Q}_\ell)$ are $\{q^{d-i}\alpha_1, \dots, q^{d-i}\alpha_n\}$. \square

In particular, this result tells us that for any variety X of dimension d , the eigenvalues of the action of φ^* on $H^i(X, \mathbb{Q}_\ell)(i)$ and $H^{2d-i}(X, \mathbb{Q}_\ell)(2d-i)$ are equal. With the addition of Lemma 3.3.7, we can now finish proving our theorem.

As discussed in 3.3.2, $\zeta(X) = \zeta(Y)$ if and only if $\text{Tr}(\varphi^* | H^i(X, \mathbb{Q}_\ell)) = \text{Tr}(\varphi^* | H^i(Y, \mathbb{Q}_\ell))$ for all $0 \leq i \leq 2d$. By Lemma 3.3.7, to prove the latter it suffices to the equalities for all $0 \leq i \leq d$. The action of φ on the 0^{th} cohomology group is trivial, implying $\text{Tr}(\varphi^* | H^0(X, \mathbb{Q}_\ell)) = \text{Tr}(\varphi^* | H^0(Y, \mathbb{Q}_\ell)) = 1$, so we may further reduce the needed values of i to those from 1 to d .

The equations (3.4) and (3.5) give a linear relation among the set of values

$$\{\text{Tr}(\varphi^* | H^i(X, \mathbb{Q}_\ell)) - \text{Tr}(\varphi^* | H^i(Y, \mathbb{Q}_\ell)) \mid i \text{ odd}, 1 \leq i \leq d\},$$

as well as a linear relation among the set of values

$$\{\mathrm{Tr}(\varphi^*|H^i(X, \mathbb{Q}_\ell)) - \mathrm{Tr}(\varphi^*|H^i(Y, \mathbb{Q}_\ell)) \mid i \text{ even}, 1 \leq i \leq d\}.$$

Hence, it is sufficient to show that $\mathrm{Tr}(\varphi^*|H^i(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^*|H^i(Y, \mathbb{Q}_\ell))$ for all but one of the even values of $1 \leq i \leq d$ (that is, $\lfloor \frac{d}{2} \rfloor - 1$ of them) and all but one of the odd values of $1 \leq i \leq d$ (that is, $\lceil \frac{d}{2} \rceil - 1$ of them). \square

Note that the hypotheses of this theorem are met in some cases where we know the varieties in question have vanishing cohomology groups. We give a few of the most prominent examples of the above theorem as corollaries:

Corollary 3.3.8. *Let X and Y be surfaces (i.e., smooth, projective varieties of dimension 2) over a finite field \mathbb{F} such that $D^b(X)$ is equivalent to $D^b(Y)$. Then X and Y have the same zeta-function.*

Proof. We go ahead and show explicitly how the proof of Theorem 3.3.6 works in the case of the even cohomology groups of surfaces. In the case where X is a surface, it has the following even Mukai–Hodge structure:

$$\tilde{H}(X, \mathbb{Q}_\ell)^{\mathrm{even}} = H^0(X, \mathbb{Q}_\ell) + H^2(X, \mathbb{Q}_\ell)(1) + H^4(X, \mathbb{Q}_\ell)(2)$$

Hence, (3.4) implies that $\mathrm{Tr}(\varphi^*|H^2(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^*|H^2(Y, \mathbb{Q}_\ell))$ in this case, and we have the equality of the traces of Frobenius acting on the even-degree cohomology groups of X and Y .

In the case of surfaces, the hypothesis of Theorem 3.3.6 does not ask for the equality of any traces of cohomology groups of odd degree because the odd Mukai–Hodge structure gives the following equality:

$$\mathrm{Tr}(\varphi^*|H^1(X, \mathbb{Q}_\ell)(1)) + \mathrm{Tr}(\varphi^*|H^3(X, \mathbb{Q}_\ell)(2)) = \mathrm{Tr}(\varphi^*|H^1(Y, \mathbb{Q}_\ell)(1)) + \mathrm{Tr}(\varphi^*|H^3(Y, \mathbb{Q}_\ell)(2))$$

Lemma 3.3.7 implies that $\mathrm{Tr}(\varphi^*|H^1(X, \mathbb{Q}_\ell)(1)) = \mathrm{Tr}(\varphi^*|H^3(X, \mathbb{Q}_\ell)(2))$ (and likewise for Y), and so we have the equality of the traces of Frobenius acting on the odd-degree cohomology groups as well. \square

The following is another special case of Theorem 3.3.6.

Corollary 3.3.9. *Let X and Y be smooth, projective varieties of dimension 3 over a finite field \mathbb{F} such that $D^b(X)$ is equivalent to $D^b(Y)$. Then X and Y have the same zeta-functions if and only if*

$$\mathrm{Tr}(\varphi^*|H^1(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(\varphi^*|H^1(Y, \mathbb{Q}_\ell)),$$

where φ is the Frobenius endomorphism.

Note that the above corollary holds for any smooth, projective varieties of dimension 3 with vanishing first cohomology group, which in particular includes Calabi–Yau 3–folds.

Chapter 4

Abelian Varieties

In this chapter, we will treat the special case of derived equivalent abelian varieties. In Section 4.1, we will prove that derived equivalent varieties over finite fields have equal zeta functions, strongly using a result of Orlov. In the Section 4.2, we will explore some results of comparing the Mukai–Hodge structures in the case of abelian varieties.

4.1 Derived equivalent abelian varieties have equal zeta functions

We first recall the following general result on derived equivalent abelian varieties:

Theorem 4.1.1 (Orlov [34, Theorem 2.19]). *Let A, B be abelian varieties over a field k . If they are derived equivalent, then there is an isometric isomorphism $A \times \widehat{A} \cong B \times \widehat{B}$.*

Definition 4.1.2. Let A, B be abelian varieties. A morphism $f : A \times \widehat{A} \rightarrow B \times \widehat{B}$ can be expressed as a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha : A \rightarrow B$, $\beta : \widehat{A} \rightarrow B$, $\gamma : A \rightarrow \widehat{B}$ and $\delta : \widehat{A} \rightarrow \widehat{B}$ are morphisms of varieties. An isomorphism $f : A \times \widehat{A} \rightarrow B \times \widehat{B}$ is called *isometric* if $\begin{pmatrix} \delta & -\widehat{\beta} \\ -\widehat{\gamma} & \alpha \end{pmatrix} : B \times \widehat{B} \rightarrow A \times \widehat{A}$ is its inverse.

Corollary 4.1.3. *Let A and B be abelian varieties defined over a field. If A and B are derived equivalent, then A and B are isogenous.*

Proof. By Poincaré’s complete reducibility theorem, abelian varieties decompose uniquely up to isogeny into products of simple abelian varieties (see Corollary 1, page 174 of [32]). Then, since any abelian variety is isogenous to its dual, $A \times \widehat{A} \cong B \times \widehat{B}$ implies that A is isogenous to B . \square

Corollary 4.1.3 holds over any field. We will be interested in the case of varieties over finite fields.

Theorem 4.1.4 (Tate [43, Theorem 1, Section 3]). *Let A and B be abelian varieties over a finite field \mathbb{F}_q . A and B are isogenous if and only if they have equal zeta functions.*

The following theorem is then a direct consequence of Corollary 4.1.3 and Theorem 4.1.4:

Theorem 4.1.5. *Let A and B be abelian varieties defined over a finite field \mathbb{F}_q . If A and B are derived equivalent, then A and B have equal zeta functions.*

To prove Theorem 4.1.5, we only need the simpler direction of Theorem 4.1.4: that isogenous varieties must have equal zeta functions, which we give a proof of below. First, recall the following definition:

Definition 4.1.6. Given an abelian variety A over an algebraically closed field and a prime p , the p -adic Tate module of A is

$$T_p(A) := \varprojlim A[p^n]$$

where $A[p^n]$ is the p^n -torsion points of A and the inverse limit is taken over maps $A[p^{n+1}] \rightarrow A[p^n]$ given by multiplication by p .

Proof of the “only if” direction of Theorem 4.1.4. Since A and B are isogenous, there is a surjection $A \rightarrow B$ whose kernel K is of finite order, so we have short exact sequence

$$0 \rightarrow K \rightarrow A \rightarrow B \rightarrow 0.$$

Let ℓ be relatively prime to q . The functor $T_\ell(-)$ from abelian varieties over \mathbb{F}_q to modules over \mathbb{Z}_ℓ given by taking the Tate module of the abelian varieties is left exact. Hence, applying the left exact functor $T_\ell(-) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ gives an exact sequence

$$0 \rightarrow T_\ell K \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \rightarrow T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \rightarrow T_\ell B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The elements of $T_\ell K$ are all torsion of order the order of K , so $T_\ell K \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, a vector space, must be 0. Hence, the map $T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \rightarrow T_\ell B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is an inclusion. The Tate modules tensored with \mathbb{Q}_ℓ are dual to the first ℓ -adic étale cohomology groups, so we have an inclusion $H^1(A, \mathbb{Q}_\ell) \rightarrow H^1(B, \mathbb{Q}_\ell)$, meaning that the characteristic polynomial f_A of the action of Frobenius φ on $H^1(A, \mathbb{Q}_\ell)$ divides the characteristic polynomial f_B of the action of Frobenius on $H^1(B, \mathbb{Q}_\ell)$.

Since A and B are isogenous, there is also a surjection $B \rightarrow A$ whose kernel is finite order, and the same argument as shown above implies that f_B also divides f_A , hence these polynomials are equal, and $\text{Tr}(\varphi^*|H^1(A, \mathbb{Q}_\ell)) = \text{Tr}(\varphi^*|H^1(B, \mathbb{Q}_\ell))$.

Once the first cohomology group of an abelian variety is known, all of its ℓ -adic étale cohomology is known: $H^i(A, \mathbb{Q}_\ell) = \bigwedge^i H^1(A, \mathbb{Q}_\ell)$ (see for instance Milne [28, Theorem 12.1(b)]), and so the eigenvalues of the action of Frobenius on the i^{th} cohomology are all products of the eigenvalues i distinct eigenvectors in $H^1(A, \mathbb{Q}_\ell)$. Thus, by the Lefschetz Fixed Point Theorem (see Section 1.3), $\zeta(A) = \zeta(B)$. \square

4.2 Mukai–Hodge structures and abelian varieties

In this section, we explore comparing the zeta functions of derived equivalent abelian varieties by using the isomorphism of their even and odd Mukai–Hodge structures. Theorem 3.3.6 in the previous chapter implies that all derived equivalent smooth, projective varieties of dimension 1 or 2 over a finite field are isogenous, without using Theorem 4.1.1. Using the method of comparing Mukai–Hodge structures, we extend the proof of Theorem 3.3.6 to the special case of 3–dimensional abelian varieties, which gives an alternate proof of Theorem 4.1.5 in the 3–dimensional case that does not rely on Theorem 4.1.1. Applying Theorem 4.1.4 then gives a proof that derived equivalent abelian varieties of dimension 3 or less are isogenous, without having used Theorem 4.1.1.

Proposition 4.2.1. *Let A and B be 3–dimensional abelian varieties over a finite field \mathbb{F}_q . If A and B have isomorphic even and odd Mukai–Hodge structures, then $\zeta(A) = \zeta(B)$.*

Proof. For any ℓ relatively prime to q , recall that the ℓ –adic étale cohomology for any abelian variety A has the following properties: $H^i(A, \mathbb{Q}_\ell) = \bigwedge^i H^1(A, \mathbb{Q}_\ell)$, and $H^1(A, \mathbb{Q}_\ell)$ is a $2d$ –dimensional \mathbb{Q}_ℓ vector space, where d is the dimension of A (see Theorem 12.1 and its proof in Milne [28]). Let $\{\alpha_1, \dots, \alpha_{2d}\}$ be the set of eigenvalues of the action of Frobenius on $H^1(A)$. Then the set of eigenvalues of the action of Frobenius on $H^i(A, \mathbb{Q}_\ell)$ is $\{\alpha_{n_1}, \dots, \alpha_{n_i}\}_{n_1 < \dots < n_i}$ (there are $\binom{2d}{i}$ of them).

We also show the following helpful claim:

Claim 4.2.2. *For any abelian variety A/\mathbb{F}_q , the eigenvalues $\{\alpha_1, \dots, \alpha_{2d}\}$ of the action of Frobenius on its degree-1 ℓ –adic étale cohomology group, for any ℓ relatively prime to q , $H^1(A, \mathbb{Q}_\ell)$, can be partitioned into complex-conjugate pairs.*

Proof of Claim. First, note that if an eigenvalue is complex (that is, in $\mathbb{C} \setminus \mathbb{R}$), then its conjugate is also an eigenvalue since the coefficients of the characteristic polynomial of Frobenius acting on ℓ –adic étale cohomology (so long as $\ell \neq q$, which we assume it is) are rational (Deligne [17, Théorème 1.6]).

Since complex eigenvalues come in pairs and $H^1(A, \mathbb{Q}_\ell)$ has an even total number of eigenvalues, there must be an even number of real eigenvalues. Any real eigenvalues of $H^1(A)$ must be $\pm q^{\frac{1}{2}}$ (Deligne [17, Théorème 1.6]), and so we can group all of them into pairs $(q^{\frac{1}{2}}, q^{\frac{1}{2}})$ and $(-q^{\frac{1}{2}}, -q^{\frac{1}{2}})$, with at most one mismatched pair left over. Since A is abelian, the eigenvalue of the action of Frobenius on $H^{2d}(A)$ is $q^d = \alpha_1 \cdots \alpha_{2d}$, and so all the eigenvalues can be matched into pairs with product q , that is, conjugate pairs. \square

Let A and B be derived equivalent varieties as in the hypothesis. Call the eigenvalues of the action of Frobenius on $H^1(A, \mathbb{Q}_\ell)$ and $H^1(B, \mathbb{Q}_\ell)$ $\{\alpha_1, \dots, \alpha_6\}$ and $\{\beta_1, \dots, \beta_6\}$, respectively. By Claim 4.2.2, we may assume, without loss of generality, that $\alpha_1\alpha_2 = \alpha_3\alpha_4 = \alpha_5\alpha_6 = \beta_1\beta_2 = \beta_3\beta_4 = \beta_5\beta_6 = q$.

The isomorphism between the odd Mukai–Hodge structures of A and B (see Lemma 3.2.4) implies that the following sets of eigenvalues are equal; these values are the roots of the characteristic polynomials of the action of Frobenius on the odd Mukai–Hodge structures.

$$\bigcup_{j=1}^2 \{\alpha_1, \dots, \alpha_6\} \cup \left\{ \frac{\alpha_{n_1} \alpha_{n_2} \alpha_{n_3}}{q} \right\}_{1 \leq n_1 < n_2 < n_3 \leq 6} = \bigcup_{j=1}^2 \{\beta_1, \dots, \beta_6\} \cup \left\{ \frac{\beta_{n_1} \beta_{n_2} \beta_{n_3}}{q} \right\}_{1 \leq n_1 < n_2 < n_3 \leq 6}$$

As a consequence of the above claim, we observe that $\left\{ \frac{\alpha_{n_1} \alpha_{n_2} \alpha_{n_3}}{q} \right\}_{1 \leq n_1 < n_2 < n_3 \leq 6}$ contains 2 copies of α_i for each i and so

$$\begin{aligned} \bigcup_{j=1}^2 \{\alpha_1, \dots, \alpha_6\} \cup \left\{ \frac{\alpha_{n_1} \alpha_{n_2} \alpha_{n_3}}{q} \right\}_{1 \leq n_1 < n_2 < n_3 \leq 6} \\ = \bigcup_{j=1}^4 \{\alpha_1, \dots, \alpha_6\} \cup \left\{ \frac{\alpha_{n_1} \alpha_{n_2} \alpha_{n_3}}{q} \right\}_{n_1 \in \{1,2\}, n_2 \in \{3,4\}, n_3 \in \{5,6\}} \end{aligned}$$

and likewise for the β_i , hence:

$$\begin{aligned} \bigcup_{j=1}^4 \{\alpha_1, \dots, \alpha_6\} \cup \left\{ \frac{\alpha_{n_1} \alpha_{n_2} \alpha_{n_3}}{q} \right\}_{n_1 \in \{1,2\}, n_2 \in \{3,4\}, n_3 \in \{5,6\}} \\ = \bigcup_{j=1}^4 \{\beta_1, \dots, \beta_6\} \cup \left\{ \frac{\beta_{n_1} \beta_{n_2} \beta_{n_3}}{q} \right\}_{n_1 \in \{1,2\}, n_2 \in \{3,4\}, n_3 \in \{5,6\}} \end{aligned} \quad (4.1)$$

Now, suppose that $\{\alpha_1, \dots, \alpha_6\} \neq \{\beta_1, \dots, \beta_6\}$. Note that since

$$\left\{ \frac{\alpha_{n_1} \alpha_{n_2} \alpha_{n_3}}{q} \right\}_{n_1 \in \{1,2\}, n_2 \in \{3,4\}, n_3 \in \{5,6\}}$$

contains 8 elements and the left-hand side of (4.1) contains 4 copies of each α_i (and likewise for β), (4.1) implies that the intersection of $\{\alpha_i\}_{1 \leq i \leq 6}$ and $\{\beta_i\}_{1 \leq i \leq 6}$ has at least four elements. Without loss of generality then, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$ and $\alpha_4 = \beta_4$, hence we may simplify (4.1) again to

$$\begin{aligned} \bigcup_{j=1}^4 \{\alpha_5, \alpha_6\} \cup \left\{ \frac{\alpha_{n_1} \alpha_{n_2} \alpha_{n_3}}{q} \right\}_{n_1 \in \{1,2\}, n_2 \in \{3,4\}, n_3 \in \{5,6\}} \\ = \bigcup_{j=1}^4 \{\beta_5, \beta_6\} \cup \left\{ \frac{\beta_{n_1} \beta_{n_2} \beta_{n_3}}{q} \right\}_{n_1 \in \{1,2\}, n_2 \in \{3,4\}, n_3 \in \{5,6\}} \end{aligned}$$

Summing all terms on each side in the above equality yields:

$$\begin{aligned} 4\alpha_5 + 4\alpha_6 + \sum_{\substack{n_1 \in \{1,2\} \\ n_2 \in \{3,4\} \\ n_3 \in \{5,6\}}} \frac{\alpha_{n_1} \alpha_{n_2} \alpha_{n_3}}{q} &= 4\beta_5 + 4\beta_6 + \sum_{\substack{n_1 \in \{1,2\} \\ n_2 \in \{3,4\} \\ n_3 \in \{5,6\}}} \frac{\beta_{n_1} \beta_{n_2} \beta_{n_3}}{q} \\ &= 4\beta_5 + 4\beta_6 + \sum_{\substack{n_1 \in \{1,2\} \\ n_2 \in \{3,4\} \\ n_3 \in \{5,6\}}} \frac{\alpha_{n_1} \alpha_{n_2} \beta_{n_3}}{q} \end{aligned}$$

If we divide each side of the above equation by $4 + \sum_{n_1 \in \{1,2\}, n_2 \in \{3,4\}} \frac{\alpha_{n_1} \alpha_{n_2}}{q}$, we are left with $\alpha_5 + \alpha_6 = \beta_5 + \beta_6$. Since α_5, α_6 and β_5, β_6 are conjugate pairs, we then have $\{\alpha_5, \alpha_6\} = \{\beta_5, \beta_6\}$. \square

Remark 4.2.3. The argument used in the proof of Proposition 4.2.1 is not sufficient to prove the same statement for 4-dimensional abelian varieties. It is not known whether Proposition 4.2.1 holds for abelian varieties of dimension greater than 3. However, if the isomorphism of the even and odd Mukai–Hodge structures of two abelian varieties does not imply that they have equal zeta functions, it would be interesting to know what information is necessary to add to the hypothesis to reach that conclusion.

Chapter 5

Isomorphism of relative Mukai–Hodge structures, and its consequences

In this chapter, we present a relative version of the work in Chapter 3.

5.1 Relative Mukai–Hodge structures

Definition 5.1.1. Let W, Z be varieties over \mathbb{F}_q . Given a morphism $a : W \rightarrow Z$, we define the *even and odd ℓ -adic étale Mukai–Hodge structures of a* to be:

$$\begin{aligned}\tilde{R}^{\text{even}}(a) &= \bigoplus_i R^{2i}a_*\mathbb{Q}_\ell(i), \\ \tilde{R}^{\text{odd}}(a) &= \bigoplus_i R^{2i-1}a_*\mathbb{Q}_\ell(i).\end{aligned}$$

Definition 5.1.2. Let $a : W \rightarrow Z$ be a map of varieties and \mathcal{F} a coherent sheaf on W . We denote by $\text{ch}_a(\mathcal{F})$ the Chern character relative to a . See also [42, Exposé 7] for some discussion of this object, but we will give a construction here.

The filtration from Leray spectral sequence for a ,

$$R^p\Gamma(\mathcal{R}^qa_*\mathbb{Q}_\ell) \Rightarrow H^{p+q}(W, \mathbb{Q}_\ell),$$

induces maps $H^i(W, \mathbb{Q}_\ell(l)) \rightarrow H^0(R^ia_*\mathbb{Q}_\ell(l))$ for any choice of i and l . $\text{ch}_a(\mathcal{F})$ is the image of $\text{ch}(\mathcal{F})$ in $\bigoplus_i H^0(R^{2i}a_*\mathbb{Q}_\ell(i))$ under these maps.

We define the relative todd class similarly:

Definition 5.1.3. Let $a : W \rightarrow Z$ be a map of varieties and \mathcal{F} a coherent sheaf on W . The relative todd class $\text{td}_a(\mathcal{F})$ is the image of $\text{td}(\mathcal{F})$ in $\bigoplus_i H^0(R^{2i}a_*\mathbb{Q}_\ell(i))$ under the maps induced by the filtration from the Leray spectral sequence $R^p\Gamma(\mathcal{R}^qa_*\mathbb{Q}_\ell) \Rightarrow H^{p+q}(W, \mathbb{Q}_\ell)$.

Weibel proves the following result as a consequence of [51, Chapter II, Theorem 6.3]:

Claim 5.1.4 ([51, Chapter II, Example 6.3.4]). *Let X be a noetherian scheme, and $i : Z \hookrightarrow X$ the inclusion of a closed subscheme. Let $\text{Coh}(Z)$ be the category of coherent \mathcal{O}_Z -modules and $\text{Coh}_Z(X)$ the category of coherent \mathcal{O}_X -modules supported on Z . There is an isomorphism of Grothendieck groups*

$$K(\text{Coh}(Z)) \cong K(\text{Coh}_Z(X))$$

induced by the pushforward i_* .

Definition 5.1.5. Let X, Y, S be smooth, projective varieties and $f : X \rightarrow S, g : Y \rightarrow S$ morphisms. Let $\Phi_P : D^b(X) \rightarrow D^b(Y)$ be a Fourier–Mukai transform with $\text{Supp}(P) \subseteq X \times_S Y$. Let ι be the inclusion $X \times_S Y \rightarrow X \times Y$. By Claim 5.1.4, there is a $Q \in D^b(X \times_S Y)$ such that $\iota_* \text{ch}(Q) = \text{ch}(P)$. We define $\text{ch}_S(P) := \text{ch}_{f \times g}(Q)$, where $f \times g : X \times_S Y \rightarrow S$.

Let $\mathcal{T}_{X \times_S Y/S}$ be the relative tangent bundle of $f \times g$, which is a sheaf on $X \times_S Y$. Then, define $\text{td}_S(\mathcal{T}_{X \times_S Y/S}) := \text{td}_{f \times g}(\mathcal{T}_{X \times_S Y/S})$.

We define $v_S(P)$ to be the Mukai vector of P relative to S :

$$v_S(P) := \text{ch}_S(P) \sqrt{\text{td}_S(\mathcal{T}_{X \times_S Y/S})} \in H^0\left(S, \bigoplus_i R^{2i}(f \times g)_* \mathbb{Q}_\ell(i)\right).$$

Definition 5.1.6. Suppose we are in the same setting as Definition 5.1.5. Then, we induce the following map from the higher direct image sheaves of $f_* \mathbb{Q}_\ell$ to those of $g_* \mathbb{Q}_\ell$:

$$\Psi_{P,S} := \pi_{Y*}(\pi_X^*(-) \cup v_S(P)), \quad (5.1)$$

where π_X and π_Y are the projection maps $X \times_S Y \rightarrow X$ and $X \times_S Y \rightarrow Y$.

Notation 5.1.7. We provide the following commutative diagram as a reference for the notation introduced in this section. Its front and back faces are pullback squares:

$$\begin{array}{ccccc} X \times_S Y & \xrightarrow{\pi_Y} & Y & & \\ \pi_X \downarrow & \searrow \iota & \downarrow g & \searrow = & \\ & X \times Y & \xrightarrow{p_2} & Y & \\ & \downarrow & \downarrow & \downarrow & \\ X & \xrightarrow{f} & S & \xrightarrow{p_1} & X \\ & \searrow = & \searrow & \searrow & \\ & & & & \text{Spec}(k) \end{array} \quad (5.2)$$

We denote $f \times g := g \circ \pi_Y = f \circ \pi_X : X \times_S Y \rightarrow S$.

Remark 5.1.8. The map $\Psi_{P,S}$, like Ψ_P , can introduce Tate twists. The map induced on the i^{th} cohomology group by the portion of (5.1) contributed by $v_S^j(P)$, the degree $2j$ part of $v_S(P)$ is as follows: Observe that, more specifically, the map in acts in the following way

$$\Psi_{P,S}^{i,j} : R^i f_* \mathbb{Q}_\ell \xrightarrow{\pi_X^*} R^i(f \times g)_* \mathbb{Q}_\ell \xrightarrow{\cup v_S^j(P)} R^{i+2j}(f \times g)_* \mathbb{Q}_\ell(j) \xrightarrow{\pi_{Y*}} R^{i+2(j-d_X)} g_* \mathbb{Q}_\ell(j-d_X), \quad (5.3)$$

where d_X is the dimension of X and d_Y is the dimension of Y . Denote the map on the i^{th} cohomology group twisted by l induced by $\Psi_{P,S}^{i,j}$ by

$$\Psi_{P,S}^{i,j}(l) : R^i f_* \mathbb{Q}_\ell(l) \rightarrow R^{i+2(j-d_X)} g_* \mathbb{Q}_\ell(l+j-d_X).$$

Observe that the following maps are well-defined:

$$\begin{aligned} \Psi_{P,S}^{\text{even}} &:= \bigoplus_{i=0}^{d_X} \sum_{j=0}^{d_X+d_Y} \Psi_{P,S}^{2i,j}(i-d) : \tilde{R}^{\text{even}}(f) \rightarrow \tilde{R}^{\text{even}}(g) \\ \Psi_{P,S}^{\text{odd}} &:= \bigoplus_{i=0}^{d_X} \sum_{j=0}^{d_X+d_Y} \Psi_{P,S}^{2i-1,j}(i-d) : \tilde{R}^{\text{odd}}(f) \rightarrow \tilde{R}^{\text{odd}}(g) \end{aligned}$$

Lemma 5.1.9. *Let $\Phi_P : D^b(X) \rightarrow D^b(Y)$ be a Fourier–Mukai equivalence smooth, projective varieties X and Y and suppose there are proper maps f and g into a third variety S such that $\text{Supp}(P)$ is contained in $X \times_S Y$. Then, the maps $\Psi_{P,S}^{\text{even}}$ and $\Psi_{P,S}^{\text{odd}}$ are isomorphisms.*

Proof. This proof follows a very similar argument to that in Lemma 3.2.4.

By Proposition 2.4.1, there is a $P' \in D^b(X \times Y)$ such that $\Phi_{P'}$ is quasi-inverse to Φ_P . Furthermore, $\Phi_{P'} \circ \Phi_P \cong \text{id}_{D^b(Y)} \cong \Phi_{\mathcal{O}_{\Delta_Y}}$ and $\Phi_P \circ \Phi_{P'} \cong \text{id}_{D^b(X)} \cong \Phi_{\mathcal{O}_{\Delta_X}}$, where \mathcal{O}_{Δ_X} and \mathcal{O}_{Δ_Y} are the pushforwards of the structure sheaves of the diagonals to $X \times X$ and $Y \times Y$.

In order to show that $\Psi_{P,S}^{\text{even}}$ and $\Psi_{P,S}^{\text{odd}}$ are isomorphisms, it suffices to prove the following two statements:

- (1) Let X, Y and Z all be varieties over S . Let $Q \in D^b(X \times Y)$, $R \in D^b(Y \times Z)$, $T \in D^b(X \times Z)$ be such that $\Phi_R \circ \Phi_Q \cong \Phi_T$ and their set-theoretic supports are contained in $X \times_S Y$, $Y \times_S Z$ and $X \times_S Z$. Then, we have $\Psi_{R,S}^{\text{even}} \circ \Psi_{Q,S}^{\text{even}} \cong \Psi_{T,S}^{\text{even}}$ and $\Psi_{R,S}^{\text{odd}} \circ \Psi_{Q,S}^{\text{odd}} \cong \Psi_{T,S}^{\text{odd}}$.
- (2) Let $\mathcal{O}_\Delta \in D^b(X \times X)$ be the pushforward of the structure sheaf of the diagonal map. Then $\Psi_{\mathcal{O}_\Delta,S}^{\text{even}}$ and $\Psi_{\mathcal{O}_\Delta,S}^{\text{odd}}$ act identically.

Notation 5.1.10. In addition to the notation from Notation 5.1.7, we will also use the following: Let $\gamma_{X,Y}$, $\gamma_{Y,Z}$ and $\gamma_{X,Z}$ be the projection maps, respectively, from $X \times_S Y \times_S Z$ to $X \times_S Y$, $Y \times_S Z$ and $X \times_S Z$. Let γ_X , γ_Y and γ_Z be the projection maps, respectively, from $X \times_S Y \times_S Z$ to X , Y and Z . Let α_Y and α_Z be the projection maps from $Y \times_S Z$ to Y and Z , respectively. Let β_X and β_Z be the projection maps from $X \times_S Z$ to X and Z , respectively.

Proof of (1): We can prove (1) in a way that is very similar to the proof that the composition of two Fourier–Mukai transforms is a Fourier–Mukai transform (compare [22, Proposition 5.10]):

$$\begin{aligned}
 \Psi_{T,s}(-) &= \beta_{Z*}(\beta_Z^*(-) \cup v_S(R)) \\
 &\cong \beta_{Z*}(\beta_Z^*(-) \cup \gamma_{X,Z*}(\gamma_{X,Y}^* v_S(Q) \cup \gamma_{Y,Z}^* v_S(R))) \\
 &\cong \beta_{Z*}(\gamma_{XZ*}(\gamma_X^*(-) \cup \gamma_{X,Y}^* v_S(Q) \cup \gamma_{Y,Z}^* v_S(R))) \quad (\text{projection formula}) \\
 &\cong \gamma_{Z*}(\gamma_{X,Y}^*(\pi_X^*(-) \cup v_S(P)) \cup \gamma_{Y,Z}^* v_S(R)) \\
 &\cong \alpha_{Z*} \gamma_{Y,Z*}(\gamma_{X,Y}^*(\pi_X^*(-) \cup v_S(Q)) \cup \gamma_{Y,Z}^* v_S(R)) \\
 &\cong \alpha_{Z*}(\gamma_{Y,Z*} \gamma_{X,Y}^*(\pi_X^*(-) \cup v_S(Q)) \cup v_S(R)) \quad (\text{projection formula}) \\
 &\cong \alpha_{Z*}(\alpha_Y^* \pi_{X*}(\pi_Y^*(-) \cup v_S(Q)) \cup v_S(R)) \\
 &\cong \Psi_{R,s}(-) \circ \Psi_{Q,s}(-)
 \end{aligned}$$

In order to prove (2), we establish the following notation:

$$\begin{array}{ccccc}
 X & & & & \\
 \Delta \searrow & & \text{id} \searrow & & \\
 & X \times_S X & \xrightarrow{\pi_2} & X & \\
 \text{id} \searrow & \downarrow \pi_1 & \searrow h & \downarrow f & \\
 & X & \xrightarrow{f} & S &
 \end{array}$$

Let $\tau(\Delta_* \mathcal{O}_X) := \text{ch}(\Delta_* \mathcal{O}_X) \text{td}(X \times_S X)$ and $\tau(\mathcal{O}_X) := \text{ch}(\mathcal{O}_X) \text{td}(X)$. Let $\tau^i(\Delta_* \mathcal{O}_X)$ denote the portion of $\tau(\Delta_* \mathcal{O}_X)$ contained in $R^{2i} h_* \mathbb{Q}_\ell(i)$, and analogously for $\tau^i(\mathcal{O}_X)$.

The following diagram commutes:

$$\begin{array}{ccccccc}
 R^j h_* \Delta_* \mathbb{Q}_\ell & \xrightarrow{\tau^i(\mathcal{O}_X)} & R^{j+2i} h_* \Delta_* \Delta^! \pi_X^! \mathbb{Q}_\ell(i)(-d_X)[-2d_X] & & & & \\
 \uparrow & & \downarrow \Delta_* \Delta^! \rightarrow \text{id} & & & & \\
 R^j f_* \mathbb{Q}_\ell & \xrightarrow{\pi_X^*} R^j h_* \mathbb{Q}_\ell & \xrightarrow{\tau^i(\Delta_* \mathcal{O}_X)} R^{j+2i} h_* \mathbb{Q}_\ell(i) & \xrightarrow{\sim} & R^{j+2i} \pi_X^! \mathbb{Q}_\ell(i)(-d_X)[-2d_X] & & \\
 & & & & \downarrow \sim & & \\
 & & & & R^{j+2i} f_* \pi_{X*} \pi_X^! \mathbb{Q}_\ell(i)(-d_X)[-2d_X] & & \\
 & & & & \downarrow \pi_{X*} \pi_X^! \rightarrow \text{id} & & \\
 & & & & R^{j+2(i-d_X)} f_* \mathbb{Q}_\ell(i-d_X) & &
 \end{array}$$

Here we use the natural isomorphism $\mathbb{Q}_\ell \rightarrow \pi_X^! \mathbb{Q}_\ell(-d_X)[-2d_X]$, defined in [1, Expose XVIII, 3.2.5], and the identification $\Delta^! \pi_X^! \simeq (\pi_X \circ \Delta_X)^! = \text{id}$. This commutative diagram, along with the fact that $\Delta^* \sqrt{\text{td}(X \times_S X)} = \text{td}(X)$ and Grothendieck–Riemann–Roch (compare [22, Proposition 5.33]) implies that $\Psi_{\mathcal{O}_{\Delta,S}}^{\text{even}}$ and $\Psi_{\mathcal{O}_{\Delta,S}}^{\text{odd}}$ act identically. \square

Theorem 5.1.11 (Application of Lemma 5.1.9). *Let X and Y be varieties over a finite field \mathbb{F}_q such that $D^b(X)$ is equivalent to $D^b(Y)$. Suppose that X , and hence also Y (see Corollary 2.3.13(c)), has a semiample canonical (or anti-canonical) bundle. Let S be the (anti-)canonical variety of X and of Y (see Proposition 2.3.3). Let $f : X \rightarrow S$ and $g : Y \rightarrow S$ be the canonical maps. Then, the maps $\Psi_{P,S}^{\text{even}}$ and $\Psi_{P,S}^{\text{odd}}$ are isomorphisms.*

Proof. The canonical morphisms are proper, and by Proposition 2.3.14, $\text{Supp}(P) \subseteq X \times_S Y$, and so this situation satisfies the hypothesis of Lemma 5.1.9. \square

5.2 Relative Mukai–Hodge structures and zeta functions

Theorem 5.2.1. *Let X and Y be smooth, projective varieties of dimension d over a finite field \mathbb{F}_q such that $D^b(X)$ is equivalent to $D^b(Y)$. Suppose that X , and hence also Y (see Corollary 2.3.13(c)), has a semiample canonical (or anti-canonical) bundle. Let S be the (anti-)canonical variety of X and of Y (see Proposition 2.3.3). If, for each geometric point $s \in S$, the fibers X_s and Y_s fulfill at least one of the following hypotheses, then $\zeta(X) = \zeta(Y)$:*

- (i) X_s and Y_s are smooth, projective varieties such that

$$\text{Tr}(\varphi^*|H^i(X_s, \mathbb{Q}_\ell)) = \text{Tr}(\varphi^*|H^i(Y_s, \mathbb{Q}_\ell))$$

for $\lfloor \frac{d}{2} \rfloor - 1$ even values and $\lceil \frac{d}{2} \rceil - 1$ odd values of $1 \leq i \leq d$, where φ is the geometric Frobenius endomorphism and $d = \dim X_s = \dim Y_s$.

- (ii) X_s and Y_s are abelian varieties of dimension 3 or lower.

Remark 5.2.2. Note that condition (i) includes the case where X_s and Y_s are surfaces or Calabi–Yau 3-folds.

Proof. By Theorem 2.2.4, the derived equivalence $D^b(X) \cong D^b(Y)$ is isomorphic to a Fourier–Mukai functor Φ_P for some $P \in D^b(X \times Y)$.

Let $f : X \rightarrow S$ and $g : Y \rightarrow S$ be the canonical maps.

By Theorem 5.1.11, the maps $\Psi_{P,S}^{\text{even}}$ and $\Psi_{P,S}^{\text{odd}}$ give isomorphisms between the even and odd ℓ -adic étale Mukai–Hodge structures of f and g :

$$\tilde{R}^{\text{even}}(f) \cong \tilde{R}^{\text{even}}(g) \quad \text{and} \quad \tilde{R}^{\text{odd}}(f) \cong \tilde{R}^{\text{odd}}(g). \quad (5.4)$$

Let $s \in S$ be a geometric point. Since f and g are proper, by the proper base change theorem (Theorem 1.3.7), given any geometric point $s \in S$, $(R^i f_*(\mathbb{Q}_\ell))_s = H^i(X_s, \mathbb{Q}_\ell)$ and $(R^i g_*(\mathbb{Q}_\ell))_s = H^i(Y_s, \mathbb{Q}_\ell)$, and so localizing the isomorphism (5.4) at s yields:

$$\begin{aligned} \bigoplus_i H^{2i}(X_s, \mathbb{Q}_\ell(i)) &\cong \bigoplus_i H^{2i}(Y_s, \mathbb{Q}_\ell(i)) \\ \bigoplus_i H^{2i-1}(X_s, \mathbb{Q}_\ell(i)) &\cong \bigoplus_i H^{2i-1}(Y_s, \mathbb{Q}_\ell(i)). \end{aligned}$$

Hence the fibers X_s and Y_s have isomorphic even and odd Mukai–Hodge structures. If X_s and Y_s satisfy condition (i) or (ii), then by Theorems 3.3.6 and Proposition 4.2.1, $\zeta(X_s) = \zeta(Y_s)$. In order to show that $\zeta(X) = \zeta(Y)$, it suffices to prove equality of zeta functions on all fibers over geometric points of s . □

Remark 5.2.3. The hypothesis of Theorem 5.2.1 only holds when, in the notation of the theorem, all fibers of X and Y over S are smooth, which is rare.

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