SIMON FRASER UNIVERSITY
Department of Economics

Econ 305 | Prof. Kasa
Intermediate Macroeconomic Theory | Spring 2012

PROBLEM SET 2
(Solutions)

1. (20 points). The following table contains recent data from the Penn World Tables (note all GDPs expressed relative to the USA):

Country	Per Capita GDP (2000)	Investment/Saving Rate
USA	1.0	.245
France	0.691	.259
Japan	0.713	.361
S. Korea	0.556	.439
Mexico	0.261	.235
China	0.183	.288
Uganda	0.027	.042

Assume that each country is described by the Solow model, with production function $Y = AK^{1/3}L^{2/3}$, and assume that each country is in a steady state. Assume that capital depreciation and labor force growth are identical across countries. (Note, you will need a calculator for this question).

(a) Assuming that productivity, A, is identical across countries, use the above data on investment rates to calculate the predicted GDP of each country (relative to the USA). How do the predictions match up to the actual data given in the first column?

Given a production function $Y = AK^{\alpha}L^{1-\alpha}$, the Solow model implies that steady state per capita income is given by

$$y = A^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Given the assumptions that $\alpha = 1/3$, and A, n, and δ are identical across countries then implies the following expression for per capita income relative to the USA

$$\frac{y_i}{y_{USA}} = \left(\frac{s_i}{s_{USA}}\right)^{1/2}$$

Substituting in the given data, we get the following predictions for relative per capita incomes:

France = 1.028 Japan = 1.214 Korea = 1.339 Mexico = .979 China = 1.084
Uganda = .414

Note that the model predicts that most of these countries should be richer than the USA! This simply follows from the fact that they save and invest more. Note, that even China is predicted to be richer. Although Uganda is predicted to be poorer than the USA, the model is still off by an order of magnitude. Clearly, something is missing.

(b) Use the model to infer what productivity must be in each country (relative to the USA). Any surprises? According to this model, which is more important in explaining cross-country income differences, investment or productivity?

One obvious suspect in the above analysis is the assumption that productivity is identical across countries. If we relax this assumption, we can use the above expressions to infer what productivity differences are across countries, conditional on the validity of the model. If we don't assume identical productivities, we get the following expressions for relative per capita incomes (in the steady state):

$$\frac{y_i}{y_{USA}} = \left(\frac{s_i}{s_{USA}}\right)^{1/2} \left(\frac{A_i}{A_{USA}}\right)^{3/2}$$

Notice that productivity has a bigger effect than saving, because it not only affects output directly, it also produces reinforcing changes in investment. Since we observe relative incomes and relative savings rates, we can use this to infer relative productivities

$$\frac{A_i}{A_{USA}} = \left(\frac{y_i}{y_{USA}}\right)^{2/3} \left(\frac{s_{USA}}{s_i}\right)^{1/3}$$

Substituting in the given data, we get the following predicitions for relative productivities:

France = .767   Japan = .701   Korea = .557   Mexico = .414   China = .305
Uganda = .162

These seem pretty plausible. Note that all countries are predicted to be less productive than the USA, especially China and Uganda.

(c) Briefly describe at least two caveats to this kind of analysis (besides the obvious one that the Solow model might be wrong!).

There are many caveats to the above analysis, the most obvious being the simple fact that the model could be wrong! Less obviously, it is important to keep in mind that the previous analysis assumed that countries were at least close to their steady states. If not, then this would likely overstate the implied productivity differences. For example, surely part of the reason that China is poor relative to the USA is the fact that it is still an 'emerging market', and is well below its long-run steady state. Another important caveat is the assumption that labor and capital inputs are identical across countries. This likely produces an upward bias to the implied productivities. Yet a third important caveat is the implicit assumption that these countries are closed economies, with no international trade or capital flows. Obviously, this misses a big part of what's going on in China!

2. (25 points). Consider a standard Solow model with two modifications: (1) Assume the production function takes the "AK" form, $Y = AK$. (As discussed in class, this might be a good approximation under a broad view of capital and/or if there are externalities associated with capital accumulation), (2) Instead of assuming that population growth is constant, assume it depends on the level of per capita income. In particular, assume that $n = n_0 \cdot k^\gamma$, where $k = K/L$ is the capital/labor ratio, n_0 is a constant, and $\gamma > -1$ is a fixed parameter. As usual, assume the saving rate, s, is constant, and that depreciation is a constant fraction, δ, of the existing capital stock.

(a) Following the usual steps, derive an expression for the steady state capital/labor ratio. Use a graph to illustrate how the economy evolves over time. (Hint: Put k on the horizontal axis, and per capita output and investment on the vertical axis). Under what conditions on γ is the steady state stable? Explain intuitively.

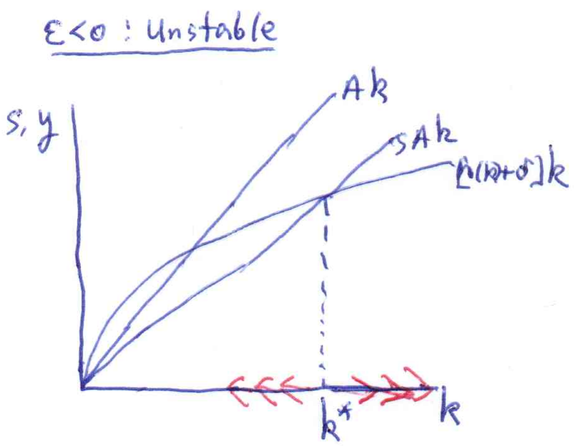
Expressed in per capita terms, we have the following steady state condition

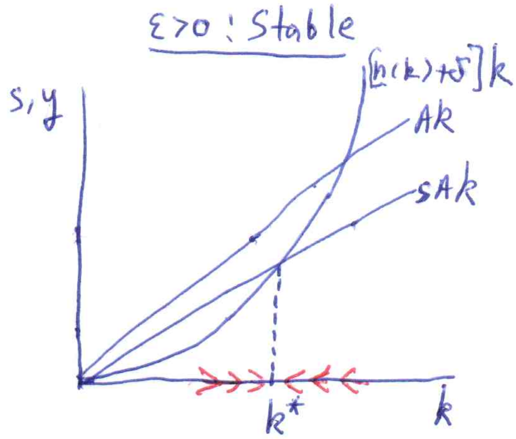
$$sAk = [n(k) + \delta] \cdot k$$

Notice that now population growth depends on income (because of its dependence on k). Given the assumption that $n(k) = n_0 k^\gamma$, we can solve for the following steady state per capita income level

$$k^* = \left(\frac{sA - \delta}{n_0}\right)^{\frac{1}{\gamma}}$$

Notice that the economy has an interior steady state, even though the production function has constant returns to k! In this model, curvature is introduced via endogenous population growth. We can visualize the economy's evolution over time with the usual Solow-type diagram, modified to account for the fact that the break-even investment line is now nonlinear. There are 2 cases to consider, depending on whether γ is positive or negative.





Notice that when $\gamma < 0$ the steady state is unstable. This is an economy where population growth declines with income. Since saving is a constant fraction of income, and there are constant returns to capital in production, if you deviate just a little beyond the steady state, per capita income will continue to keep growing forever, simply because population growth is slowing down. On the other hand, notice that if $\gamma > 0$, then the steady state is stable. This reflects the Malthusian notion that population growth increases when the economy becomes wealthier. In this case, the economy effectively has diminishing returns, even though production features constant returns. (The above graphs make the reasonable assumption that $sA > \delta$. This guarantees that in the case $\gamma < 0$, as k gets bigger, eventually the slope of the breakeven line lies below the slope of the saving function, as drawn. It also ensures that when k is really small, the slope of the breakeven line is less than the saving function when $\gamma > 0$.)

(b) How does per capita income respond to an increase in the saving rate, s? How does your answer depend on γ? Explain.

When s increases, the saving function becomes steeper (ie., rotates counter-clockwise). It is visually obvious from the previous graphs that this produces a decrease in steady state income when *the steady state is unstable (ie, when $\gamma < 0$. On the other hand, when the economy is stable, an increased saving rate produces the anticipated effect of increasing per capita income. In both cases, for there to be a new steady state an increased saving rate requires an increase in the breakeven level of investment. When population growth declines with income, this requires the economy to become poorer, so that population growth increases. Conversely, when population growth increases with income, a higher saving rate requires a higher steady state capital/labor ratio, so that population growth again rises to match the higher saving rate.*

3. (25 points). Consider a standard one-sector Solow model with a fixed savings rate s. Output is produced via the production function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

As usual, labor is inelastically supplied and grows at the exogenous rate n (i.e., $L_{t+1} = (1+n)L_t$).

Suppose that each unit of output produced generates Ω_t units of 'pollution', and that due to exogenous technological progress in pollution abatement, Ω_t decreases over time at rate g_a (i.e., $\Omega_{t+1}/\Omega_t = 1/(1+g_a)$). In addition, suppose that there is an 'abatement technology' that allows resources to be diverted into pollution reduction. Specifically, if θ represents the share of output used in pollution reduction, then net pollution emission, E_t, is given by

$$E_t = a(\theta)\Omega_t Y_t \tag{1}$$

where $a(\theta)$ is assumed to be a positive, decreasing function. For simplicity, assume that θ is constant and exogenous.

As usual, for notational convenience, let y_t represent net output available for consumption and capital accumulation per capita. That is,

$$y = \frac{(1-\theta)Y}{L}$$

Similarly, let k and e be capital and net pollution emission per capita. Using this notation, we have the following 'green Solow' model:

$$\begin{aligned} y_t &= (1-\theta)k_t^{\alpha} \\ \Delta k_{t+1} &= s(1-\theta)k_t^{\alpha} - (\delta+n)k_t && (2) \\ e_t &= a(\theta)\Omega_t k_t^{\alpha} && (3) \end{aligned}$$

where δ is the depreciation rate of capital. Evidently, from equation (2), the economy will converge to a unique steady state $k_t = k^*$. From equation (3), it is then clear that in the steady state (ie, when k_t is constant), pollution grows at a constant rate, g_E, given by

$$g_E = \frac{\Delta E_{t+1}}{E_t} = \frac{1+n}{1+g_a} - 1 \approx n - g_a$$

Of course, during the *transition* to the steady state, pollution may be growing either faster or slower than this. Now, let's define 'sustainable' growth to be a situation where $g_a \geq n$. That is, pollution remains bounded.

(a) It is often claimed that the time path of pollution within economies follows a so-called 'Environmental Kuznets Curve' (EKC), with pollution rising as the economy develops, and then eventually falling once the economy becomes wealthy enough (i.e., it traces out an inverted U-shape when plotted against either time or per capita income). Consider a sustainable economy, where $g_E < 0$. Under what conditions will this economy feature an EKC? (Hints: (1) Derive expressions for

$\Delta k_{t+1}/k_t$ and $\Delta E_{t+1}/E_t$ as functions of k_t during the transition to the steady state, (2) The growth rate of $\Omega_t k_t^\alpha$ can be approximated by $\frac{\Delta\Omega_{t+1}}{\Omega_t} + \alpha\frac{\Delta k_{t+1}}{k_t}$). If you can't provide explicit analytical conditions, then at least try to explain intuitively how this relationship could arise.

From eq. (2), we have the following expression for the (percentage) change in k_t during the transition to the steady state

$$\frac{\Delta k_{t+1}}{k_t} = s(1-\theta)k^{\alpha-1} - (\delta + n)$$

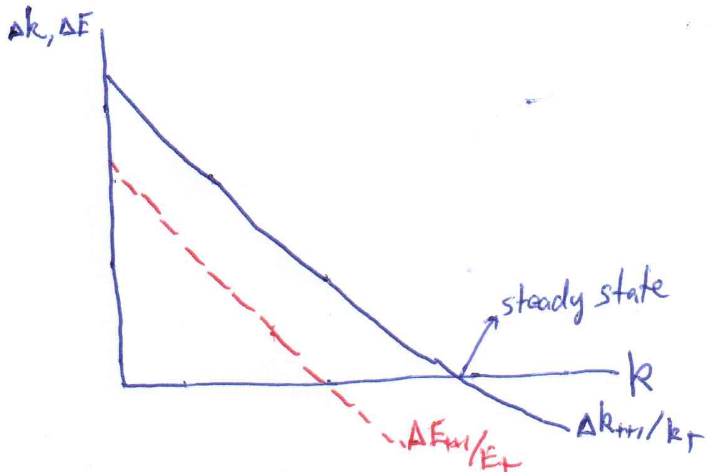
Note that this is a declining function of k, which is positive for small values of k, and negative for large values of k. The steady state occurs when it crosses the horizontal axis. Next, by definition, we have the following expression for the (percentage) change in pollution emissions during the transition to the steady state

$$\frac{\Delta E_{t+1}}{E_t} = \frac{\Omega_{t+1}Y_{t+1} - \Omega_t Y_t}{\Omega_t Y_t}$$

Using the given hints and the previous results, we can write this also as function of k as follows

$$\frac{\Delta E_{t+1}}{E_t} = \alpha(1+n)[s(1-\theta)k_t^{\alpha-1} - (\delta+n)] + n - g_a$$

Note that this has the same basic shape as the capital accumulation function, but it crosses the horizontal axis before it. This is because of the fact that when the economy is 'sustainable' $n < g_a$. In sum, we can describe the paths of capital and pollution in the following picture



Now, notice that if the economy is initially poor, with a relatively low value k, then during the transition both output and pollution increase as the economy develops. Then, since the $\Delta E_{t+1}/E_t$ line hits zero first, pollution begins to decline, while per capita income continues to grow. This would produce an Environmental Kuznets Curve. However, if the economy starts out relatively rich, meaning that its initial k is to the right of the point where $\Delta E_{t+1} = 0$ but to the left of the steady state, the economy continues to grow, but pollution declines monotonically. In this case, there would be no EKC. The intuition here is that due to diminishing returns, output growth is faster in poor countries. At the same time, since pollution growth is just proportional to output growth, pollution emissions grow rapidly in poor countries. The good news is that 'productivity' in pollution abatement is constant, and independent of the level of income. Eventually, productivity growth in pollution abatement comes to dominate the output effect, since output growth declines over time.

(b) How does an increase in abatement effort (i.e., an increase in θ) affect the time path of pollution? Explain intuitively, and relate your conclusions to how a standard Solow model reacts to an increase in the savings rate. (Hint: You do not need to solve for anything. Just sketch out a time path).

In this model, changes in 'abatement effort', $a(\theta)$ are just like changes in the saving rate in the regular Solow model. They produce temporary changes in the growth rate of pollution, but not long lasting changes. Those are determined entirely by the underlying growth rates of output and abatement productivity. The picture would look just like the economy's response to a (permanent) change in the saving rate. There would be a sudden drop in the growth rate of pollution, but it would eventually climb back to its long-run balanced growth path.