Answer to PS1/Q2: The Discrete-Time LQR

a.) <u>The Euler Eq. Approach</u> (Sub constraint into objective)

Define $L(x_t, x_{t+1}) = \frac{1}{2}\left[q x_t^2 + \frac{r}{b^2}(x_{t+1} - a x_t)^2\right]$

Then (locally) objective can be written,

$$L(x_{t-1}, x_t) + \beta L(x_t, x_{t+1})$$

<u>FOC</u>: $L_2 + \beta L_1 = 0$ Euler Eq. (subscripts = partial derivatives)

$$L_2 = \frac{r}{b^2}(x_t - a x_{t-1})$$

$$L_1 = q x_t - \frac{r}{b^2}(x_{t+1} - a x_t)$$

This gives us the following <u>2nd-order</u> <u>linear</u> difference eq,

$$\boxed{x_{t+1} - \left(a + \frac{1}{\beta a} + \frac{qb^2}{ra}\right) x_t + \frac{1}{\beta} x_{t-1} = 0}$$

The roots are given by the <u>characteristic eq</u>; $\lambda^2 - \left(a + \frac{1}{\beta a} + \frac{qb^2}{ra}\right)\lambda + \frac{1}{\beta} = 0$

Note, can write this as, $(\lambda - \phi_1)(\lambda - \phi_2)$ where $\boxed{\phi_1 \phi_2 = \frac{1}{\beta}}$

Hence, we know one root is β-stable and one is β-unstable

From quad. formula, stable root is

$$\phi = \frac{\left(a + \frac{1}{\beta a} + \frac{qb^2}{ra}\right) - \sqrt{\left(a + \frac{1}{\beta a} + \frac{qb^2}{ra}\right)^2 - 4/\beta}}{2}$$

<u>Soln</u>: $\boxed{x_t = x_0 \phi^t}$ } Non-recursive rep. of optimal path

b.) <u>The Lagrangian/Hamiltonian Approach</u> (see pages 138-42 in Ljungqvist & Sargent)

Objective: $\sum \beta^t \{ \frac{1}{2}(q x_t^2 + r u_t^2) + \lambda_{t+1}(a x_t + b u_t - x_{t+1}) \}$

$\uparrow$ co-state

<u>FOCs</u>

u_t: $r u_t + b \lambda_{t+1} = 0$

x_t: $q x_t + a \lambda_{t+1} - \beta^{-1} \lambda_t = 0$

Use constraint to sub out u_t, and write

$$\begin{pmatrix} 1 & b^2/r \\ 0 & a \end{pmatrix} \begin{pmatrix} x_{t+1} \\ \lambda_{t+1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ -q & 1/\beta \end{pmatrix} \begin{pmatrix} x_t \\ \lambda_t \end{pmatrix}$$

Invert L.H.S matrix,

$$\begin{pmatrix} x_{t+1} \\ \lambda_{t+1} \end{pmatrix} = \begin{pmatrix} a + \frac{b^2 q}{ar} & -\frac{b^2}{\beta a r} \\ -q/a & \frac{1}{\beta a} \end{pmatrix} \begin{pmatrix} x_t \\ \lambda_t \end{pmatrix} = A \begin{pmatrix} x_t \\ \lambda_t \end{pmatrix}$$

Eigenvalues solve the char. eq. $|A - \cancel{\lambda} I| = 0$

$$\left(a + \frac{b^2 q}{ar} - \phi\right)\left(\frac{1}{\beta a} - \phi\right) - \frac{q b^2}{\beta r a^2} = 0$$

On $\phi^2 - \phi\left(\frac{1}{\beta a} + a + \frac{b^2 q}{ar}\right) + 1/\beta = 0$

Same as before!

c.) Dynamic Programming (see pgs. 130-132 in Ljungvist & Sargent)

Bellman Eq.: $V_t = \min_u \left\{ \frac{1}{2}(qx_t^2 + ru_t^2) + \beta V_{t+1}(X_{t+1}) \right\}$

Guess: $V_t = \frac{1}{2} P_t X_t^2 \Rightarrow u_t = \frac{-\beta a b P_{t+1}}{r + \beta b^2 P_{t+1}} X_t$

sub in to Bellman & cancel the common X_t^2 terms

$$P_t = q + \beta a^2 P_{t+1} - \frac{\beta^2 a^2 b^2 P_{t+1}^2}{r + \beta b^2 P_{t+1}}$$

} Riccati Eq. (see 5.2.9 in Ljung & Sargent note different notation),

$$= q + \frac{\beta r a^2 P_{t+1}}{r + \beta b^2 P_{t+1}}$$

Under the conds. discussed in LS, (e.g., controllability), this has a unique positive, steady state, soln.

The optimal path of X_t is now given recursively,

$$X_{t+1} = \left(a - \frac{\beta a b^2 P}{r + \beta b^2 P}\right) X_t = \left(\frac{ar}{r + \beta b^2 P}\right) X_t$$

Using the Ric. Eq. & Char. Eq., can show $\frac{ar}{r + \beta b^2 P} = \phi$

Also can show that the Lag. Mult. is related to value function as follows

$$\boxed{\lambda_{t+1} = P X_{t+1}}$$

λ = Marg. Value of State

See p. 135 in LS