

PROBLEM SET 3 - Growth  
(Due November 19)

1. Consider the following economy. There is a representative agent who has preferences given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where  $u$  is differentiable, increasing, and strictly concave. Technology is given by

$$\begin{aligned} c_t + x_t + g_t &\leq f(k_t, g_t) \\ k_{t+1} &\leq (1 - \delta)k_t + x_t \end{aligned}$$

along with the initial condition,  $k_0 > 0$ . Here  $k_t$  and  $g_t$  are capital per worker and government spending per worker. The production function,  $f$ , is strictly concave, twice differentiable, and satisfies the usual Inada conditions.

- (a) Derive a set of equations that characterize an interior solution to the planner's problem when the planner can choose the sequence of government spending.
  - (b) Derive the equations that characterize a steady state.
  - (c) Describe how the steady state levels of capital per worker and government spending per worker vary as a function of the discount factor  $\beta$ . Interpret the results.
  - (d) Now assume that the level of technology can vary. Specifically, assume that the production function takes the form:  $f(k, g, z) = zk^\alpha g^\eta$ , where  $0 < \alpha < 1$ ,  $0 < \eta < 1$ , and  $\alpha + \eta < 1$ . Go as far as you can in describing how the steady state investment/GDP and government spending/GDP ratios vary with the technology level  $z$ . Interpret the results.
2. Consider the following continuous-time growth model with both physical capital,  $K(t)$ , and human capital,  $H(t)$ :

$$\begin{aligned} Y(t) &= [(1 - a_K)K(t)]^\alpha [(1 - a_H)H(t)]^{1-\alpha}, & 0 < a_K < 1 & \quad 0 < a_H < 1 \\ \dot{K}(t) &= sY(t) - \delta_K K(t) \\ \dot{H}(t) &= B[a_K K(t)]^\gamma [a_H H(t)]^\phi [A(t)L(t)]^{1-\gamma-\phi} - \delta_H H(t), & \gamma > 0 & \quad \phi > 0 & \quad \gamma + \phi < 1 \\ \dot{L}(t) &= nL(t) \\ \dot{A}(t) &= gA(t) \end{aligned}$$

where  $a_K$  and  $a_H$  are the fractions of the stocks of physical and human capital used in the ‘education sector’,  $\delta_K$  and  $\delta_H$  are the depreciation rates of physical and human capital, and  $B$  is a proportionality constant.

Notice that this model assumes that human capital is produced in its own sector with its own production function. Raw labor ( $L$ ) is useful only as something to be educated, not as an input into the production of final goods. Similarly, knowledge ( $A$ ) is useful only as something that can be conveyed to students, not as a direct input to goods production.

- (a) Define  $k = K/(AL)$  and  $h = H/(AL)$ . Derive equations for  $\dot{k}$  and  $\dot{h}$ .
- (b) Derive an equation describing the set of combinations of  $h$  and  $k$  such that  $\dot{k} = 0$ . Sketch it in  $(h, k)$  space. Do the same for  $\dot{h} = 0$ .
- (c) Does this economy have a balanced growth path? If so, is it unique? Is it stable? What are the growth rates of per capita output, physical capital, and human capital?
- (d) Suppose the economy is initially on a balanced growth path, and that there is a permanent increase in  $s$ . How does this affect the path of per capita output over time? Explain.

3. Consider the baseline Cass-Koopmans model,

$$\max_c \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

subject to

$$c_t + k_{t+1} = z_t k_t^\alpha + (1 - \delta)k_t$$

- (a) Derive an expression for the steady state capital/labor ratio,  $\bar{k}$ . Use a phase diagram to illustrate the economy’s dynamic adjustment to the steady state, with  $k_t$  on the horizontal axis, and  $c_t$  on the vertical axis.
- (b) Now let’s use DYNARE to simulate the economy’s responses to various productivity shocks. Assume that initially the economy is in a steady state with  $z = 1.0$ . Run the program CASSKOOP1.MOD in Matlab to simulate the response to a one-time shock at  $t = 0$ , where  $z_0 = 1.2$ , and then returns back to  $z_t = 1.0$  for  $t \geq 1$ . Assume  $\beta = 0.98$ ,  $\alpha = 0.5$ , and  $\delta = 0.98$ . For the first simulation set  $\gamma = 1.0$ , and then re-do it with  $\gamma = 5.0$ . Compare the two results and explain their differences. (Hints: (1) To run dynare in matlab, just type ”dynare casskoop1” at the matlab prompt. (2) If there is a kink at the end of simulation, try increasing the number of periods).
- (c) Now assume the productivity shock is permanent. That is,  $z_t = 1.2$  for all  $t \geq 0$ . This can be done by running the Dynare program CASSKOOP2.MOD. Again, compare the results for  $\gamma = 1.0$  and  $\gamma = 5.0$ . Which case exhibits faster convergence to the new steady state. Why? (Again: If there is a kink at the endpoint, try increasing the number of periods).

- (d) Finally, now assume productivity is stochastic. In particular, assume the log of  $z_t$  follows the AR(1) process  $z_t = \rho z_{t-1} + \sigma \varepsilon_t$ , where  $\rho = 0.96$  and  $\sigma = 0.01$ . Set  $\gamma = 2.0$ , and assume all the other parameters are the same as before. Use the Dynare program CASSKOOP3.MOD to conduct a stochastic simulation of this model. Provide a table of auto- and cross-correlations in consumption, capital, and productivity. Provide graphs of the impulse response functions.
4. Now let's use dynamic programming to solve a standard growth model. Consider an infinitely-lived representative household that owns a stock of capital,  $k$ , which it rents to a competitive firm. Capital depreciates at rate  $\delta$ . The household does not value leisure, and supplies one unit of labor inelastically. Preferences over consumption take the standard time-additive form, with discount factor  $\beta$ . The production function is Cobb-Douglas,  $Y_t = z_t K_t^\alpha N_t^{1-\alpha}$ , where  $z_t$  denotes (exogenous) total factor productivity, and  $K_t$  denotes the aggregate capital stock. (Note, in equilibrium,  $k_t = K_t$  and we can normalize  $N = 1$ ).
- (a) Write a Matlab program that solves the deterministic version of this model, by setting  $z_t = 1 \forall t$ . Set the parameters as follows:  $\beta = .95$ ,  $\delta = .10$ ,  $u(c) = c^{1-\gamma}/(1-\gamma)$ , with  $\gamma = 3$ , and  $\alpha = .3$ . Two Hints: (1) Exploit the 2nd Welfare Theorem and compute equilibrium quantities by solving a planner's problem, (2) Compute the optimal policy function by first iterating on the planner's Bellman Equation. Make sure the capital stock grid includes the steady state, by first solving for the steady state. Plot the the policy function,  $K_{t+1} = g(K_t)$ , with  $K_{t+1}$  on the vertical axis and  $K_t$  on the horizontal axis. Also plot a 45 degree line so as to exhibit the steady state capital stock (i.e., where  $K_{t+1} = K_t$ ).
- (b) Provide a sketch of the saddlepath with  $\Delta c$  on the vertical axis and  $\Delta K$  on the horizontal axis. Now compute and plot the numerical saddlepath (i.e., compute the function  $c_t = h(K_t)$  using the optimal policy function for  $K_{t+1}$  and the resource constraint).
- (c) Re-do the above calculations with  $\gamma = 1.01$ . Compare the two saddlepaths. Interpret the results.