Topic 3: Risk Sharing and International Financial Markets

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Contents

3.1 Trade across random states of nature

These notes outline a few basic models of trade in assets, with representative agents and when there are no frictions on the good markets. We start by studying complete asset markets and then look at specific cases of market incompleteness. In subsequent lectures, after reviewing the empirical evidence, we will study models with frictions on the good markets, asset markets, as well as models with endogenous incompleteness of asset markets.

3.1.1 Complete Markets, two periods

Small country; two periods, t and $t+1$; a single traded good; two states of nature.

- Representative household.

  Endowment at date $t = Y_t$. Output at date $t+1$ is uncertain: $Y_{t+1}(s)$ with probability $\pi(s)$.

  Agents make contingency plans for consumption. We will take here a simple example with only 2 states of nature (generalization straightforward).

  $$U = u(C_t) + \beta\left[\pi(1)\, u(C_{t+1}(1)) + \pi(2)\, u(C_{t+1}(2))\right]$$

  Note that the utility itself does not depend on the state of nature. $\pi(1) + \pi(2) = 1$.

- Arrow-Debreu security:

  The owner of a "state s" Arrow Debreu security gets 1 unit of output if that state of the world occurs and 0 otherwise. We assume there is a competitive market for Arrow-Debreu securities for every state s.

  A riskless asset (bond) can be interpreted as a linear combination of Arrow-Debreu securities.

  When people can trade Arrow-Debreu securities corresponding to every future state of nature, the economy has complete asset markets (complete spanning of the states of the world).

Remark 1 An interesting question within the complete market framework is to ask under what circumstances one can replicate the complete market economy with a smaller number of assets. For instance, if there is no aggregate risk a riskfree bond may be sufficient to spread risk efficiently.

- Budget constraint:

  We call $B_{t+1}(s)$ the stock of state s Arrow-Debreu securities the representative agent holds at the beginning of period $t+1$. $q(s)$ is the price at date t of such a security in terms of date t consumption, i.e. a claim to one output unit to be delivered on date t+1 if, and only if state s occurs.

$$q(1)\,B_{t+1}(1) + q(2)\,B_{t+1}(2) = Y_t - C_t$$
$$C_{t+1}(1) = Y_{t+1}(1) + B_{t+1}(1)$$
$$C_{t+1}(2) = Y_{t+1}(2) + B_{t+1}(2)$$

The intertemporal budget constraint is therefore:

$$C_t + q(1)\,C_{t+1}(1) + q(2)\,C_{t+1}(2) = Y_t + q(1)\,Y_{t+1}(1) + q(2)\,Y_{t+1}(2)$$

Remark 2 Arrow-Debreu securities transfer purchasing power across time and states. They enable consumption smoothing across states.

Remark 3 This is different from the usual constraint: there is a unique budget constraint and consumption and income are state dependent.

- Utility optimization

  Replace C_t in the utility function by its expression and maximise with respect to the bonds. We get the first order conditions (for $s = 1, 2$):

$$q(s) = \frac{\pi(s)\,\beta u'(C_{t+1}(s))}{u'(C_t)}$$

3.1.1.1 A few implications

- We could construct a riskless bond with the two securities and by arbitrage, the prices would have to be such that:

$$q(1) + q(2) = \frac{1}{1+r}$$

  so the FOC give:

$$(q(1) + q(2))\,u'(C_t) = \pi(1)\,\beta u'(C_{t+1}(1)) + \pi(2)\,\beta u'(C_{t+1}(2))$$

equivalent to:

$$u'(C_t) = (1+r)\,\beta E_t\left[u'(C_{t+1})\right]$$

- This is just the usual stochastic Euler equation. Same interpretation: the expected MRS is equal to the relative price of period 2 consumption. However, with complete markets there is much more smoothing (across states and periods) than with a risk free asset (across periods).

- The FOC can also be written as:
$$\frac{q(1)}{q(2)} = \frac{\pi(1) u'(C_{t+1}(1))}{\pi(2) u'(C_{t+1}(2))}$$

- It is optimal to equate consumption at date $t+1$ in the 2 states only when
$$\frac{q(1)}{q(2)} = \frac{\pi(1)}{\pi(2)}$$
which implies that $q(s) = \pi(s) / (1+r)$.

  In that case the Arrow Debreu securities are "*actuarially fair*". At actuarially fair prices, a country trading in complete asset markets will fully insure against all future consumption fluctuations.

- If agents are risk neutral, we also obtain that:
$$\frac{q(1)}{q(2)} = \frac{\pi(1)}{\pi(2)}$$
and also $(1+r)\beta = 1$. In that case, agents are indifferent as to whether they consume now or tomorrow, in state 1 or in state 2.

- Suppose now that the economy is under autarky, with CRRA preferences. This implies that:
$$\frac{q(1)}{q(2)} = \frac{\pi(1)}{\pi(2)} \left(\frac{Y_{t+1}(1)}{Y_{t+1}(2)}\right)^{-\rho}$$
so that the Arrow Debreu prices are not fair unless $Y_{t+1}(1) = Y_{t+1}(2)$, i.e. unless there is no risk.

- We can rewrite the FOC as:
$$\frac{u'(C_{t+1}(1))}{u'(C_{t+1}(2))} = \frac{q(1)}{q(2)} \frac{\pi(2)}{\pi(1)}$$
The RHS is independent of the agent we consider: this implies that the marginal rates of substitution between state 1 and 2 are equalized for all agents. More generally, the FOC also implies:
$$\frac{q(s)}{\pi(s)} = \frac{\beta u'(C_{t+1}(s))}{u'(C_t)}$$
so that the MRS between consumption at t and consumption at $t+1$ in state s is equal across agents/countries.

3.1.2 A log-utility case.

Suppose that $u(c) = \ln(c)$ so that
$$U = \ln(C_t) + \beta [\pi(1) \ln(C_{t+1}(1)) + \pi(2) \ln(C_{t+1}(2))]$$

- define the wealth of the country in period t
$$W_t = Y_t + q(1) Y_{t+1}(1) + q(2) Y_{t+1}(2)$$

- We know that consumption is simply:
$$C_t = \frac{1}{1+\beta} W_t$$

Similarly,

$$C_{t+1}(s) = \frac{\beta}{1+\beta}\frac{\pi(s)}{q(s)}W_t$$

from this we see that the current account is period 1 is:

$$CA_t = Y_t - C_t = \frac{\beta}{1+\beta}Y_t - \frac{1}{1+\beta}\left[q(1)\,Y_{t+1}(1) + q(2)\,Y_{t+1}(2)\right]$$

Assuming that $\beta(1+r) = 1$, we can rewrite this as a function of the period 2 income, in terms of certain period 2 consumption, as:

$$\begin{aligned} CA_t &= \frac{\beta}{1+\beta}\left[Y_t - \left(p(1)\,Y_{t+1}(1) + p(2)\,Y_{t+1}(2)\right)\right] \\ &= \frac{\beta}{1+\beta}\left[Y_t - \bar{Y}_{t+1}\right] \end{aligned}$$

where $p(s) = q(s)(1+r)$ is the price of the AD security for state s in terms of state s good, and $\bar{Y}_{t+1} = p(1)\,Y_{t+1}(1) + p(2)\,Y_{t+1}(2)$. So the current account is in surplus or deficit according to whether current income exceeds or not second period income, valued using AD prices.

3.1.2.1 Autarky interest rates and the current account (see Svensson 1988 AER)

The simple bond only economy intuition in terms of comparative advantage does not apply as simply under complete markets. Recall that we could define the autarky interest rate r^A. A country would borrow whenever $r^A > r$ and lend in the opposite case.

We can define the autarky interest rate r^A as the interest rate that would obtain if we close all financial flows:

$$\begin{aligned} \frac{1}{1+r^A} &= \frac{\beta E_t\left(u'(Y_{t+1})\right)}{u'(Y_t)} \\ &= \beta Y_t E_t\left(\frac{1}{Y_{t+1}}\right) \\ &= \beta Y_t\left(\frac{\pi(1)}{Y_{t+1}(1)} + \frac{\pi(2)}{Y_{t+1}(2)}\right) \end{aligned}$$

where the second line assumes log utility.

Alternatively, we can define the real interest rate that would prevail if no intertemporal borrowing were allowed, yet trade in state contingent claims was allowed. This is equivalent to imposing

$$\begin{aligned} C_t &= Y_t \\ q(1)\,C_{t+1}(1) + q(2)\,C_{t+1}(2) &= \frac{\bar{Y}_{t+1}}{1+r} \end{aligned}$$

Optimal choice of $C_{t+1}(s)$ imposes:

$$C_{t+1}(s) = \frac{\pi(s)}{q(s)}\frac{\bar{Y}_{t+1}}{1+r}$$

so that

$$\begin{aligned} \frac{1}{1+r^{CA}} &= \frac{\beta E_t\left(u'(C_{t+1})\right)}{u'(Y_t)} \\ &= \beta Y_t\left(\frac{\pi(1)}{C_{t+1}(1)} + \frac{\pi(2)}{C_{t+1}(2)}\right) \\ &= \beta Y_t\left(\frac{p(1)}{\bar{Y}_{t+1}} + \frac{p(2)}{\bar{Y}_{t+1}}\right) \\ &= \beta\frac{Y_t}{\bar{Y}_{t+1}} \end{aligned}$$

so that

$$\begin{aligned} CA_t &= \frac{\beta}{1+\beta}\left[Y_t - \bar{Y}_{t+1}\right] \\ &= \frac{\beta}{1+\beta}Y_t\left[1 - \frac{\bar{Y}_{t+1}}{Y_t}\right] \\ &= \frac{\beta}{2+r}Y_t\left[r - r^{CA}\right] \end{aligned}$$

so that what drives the current account is the difference between the world real interest rate and the current account autarky real interest rate. This implies that the size of the current account only depends upon the size of the gains that can be achieved from *intertemporal* re-allocation of consumption, not the reallocation *across states*.

This simple model also allows to solve for the gross capital flows.

The demand for state s asset is:

$$\begin{aligned} B_{t+1}(s) &= C_{t+1}(s) - Y_{t+1}(s) \\ &= \frac{\beta}{1+\beta}\frac{\pi(s)}{q(s)}W_1 - Y_{t+1}(s) \\ &= \frac{\beta}{1+\beta}\frac{\pi(s)}{q(s)}\left[Y_t + \frac{\bar{Y}_{t+1}}{1+r}\right] - Y_{t+1}(s) \end{aligned}$$

This gross flow can be quite large, even if the current account is small. For instance, when $r^{CA} = r$ (that is $Y_t = \bar{Y}_{t+1}$) the current account is zero. In that case,

$$B_{t+1}(s) = \beta\frac{\pi(s)}{q(s)}Y_t - Y_{t+1}(s)$$

and the autarky price of the AD security for state s satisfies:

$$q^A(s) = \pi(s)\frac{\beta Y_t}{Y_{t+1}(s)}$$

so that:

$$\begin{aligned} B_{t+1}(s) &= \beta\frac{\pi(s)}{q(s)}Y_t\left[1 - \frac{Y_{t+1}(s)\,q(s)}{Y_t\pi(s)\,\beta}\right] \\ &= \beta\frac{\pi(s)}{q(s)}Y_t\left[1 - \frac{q(s)}{q^A(s)}\right] \end{aligned}$$

and the gross flows can be much larger than the net flows (zero in that specific case). Note also the sign, consistent with usual trade theory:

- a country imports ($B_{t+1} > 0$) a security when the autarky price is high compared to the world price $\left(q^A > q\right)$.
- a country exports a security ($B_{t+1} < 0$) when the autarky price is lower than the world price $\left(q^A > q\right)$.

In general, the mapping from the riskfree rate to the direction of the current account is less easy: this is because it is de facto a model with more than 2 goods. With S states and two periods, there are $S + 1$ dated goods.

3.1.3 A two-country model

- The 2 countries are exactly symmetric.

- Good markets equilibrium (ressource constraint):

$$C_t + C_t^* = Y_t + Y_t^*$$

$$C_{t+1}(s) + C_{t+1}^*(s) = Y_{t+1}(s) + Y_{t+1}^*(s)$$

for all s.

- Denote total world output by:

$$Y^w = Y + Y^*$$

Let's write the maximization problem for country j under complete markets:

$$\begin{aligned} & \max_{C_t^j, C_{t+1}^j(s)} EU^j \\ s.t. \sum_s q(s) C_{t+1}^j(s) + C_t^j &= \sum_s q(s) Y_{t+1}^j(s) + Y_t^j \end{aligned}$$

the first order conditions are

$$\begin{aligned} \beta u'\left(C_{t+1}^j(s)\right)\pi(s) &= q(s)\mu_j \\ u'\left(C_t^j\right) &= \mu_j \end{aligned}$$

where μ_j is the LM associated with the budget constraint for country j. This implies that

$$C_{t+1}^j(s) = u'^{-1}\left(\frac{\mu_j}{\mu_i} u'\left(C_{t+1}^i(s)\right)\right)$$

and summing over all countries and using the ressource constraint:

$$Y_{t+1}(s) = \sum_j C_{t+1}^j(s) = \sum_j u'^{-1}\left(\frac{\mu_j}{\mu_i} u'\left(C_{t+1}^i(s)\right)\right)$$

This implies that $C_{t+1}^i(s)$ is only a function of the aggregate endowment $Y_{t+1}(s)$ and not of the history of previous consumptions/endowments....

- For a CRRA utility function, the Euler equation and the good market equilibrium imply:

$$q(s) = \pi(s)\beta\left(\frac{C_{t+1}(s)}{C_t(s)}\right)^{-\rho}$$

or

$$C_{t+1}(s) = C_t(s)\left(\frac{q(s)}{\pi(s)\beta}\right)^{-1/\rho}$$

summing over countries and using the world ressource constraint:

$$Y_{t+1}^w(s) = Y_t^w(s)\left(\frac{q(s)}{\pi(s)\beta}\right)^{-1/\rho}$$

or:

$$q(s) = \pi(s)\beta\left[\frac{Y_{t+1}^w(s)}{Y_t^w}\right]^{-\rho}$$

so that:

$$\frac{q(s)}{q(s')} = \frac{\pi(s)}{\pi(s')}\left[\frac{Y_{t+1}^w(s)}{Y_{t+1}^w(s')}\right]^{-\rho}$$

- This implies that AD prices will be fair if and only if there is no aggregate risk: $Y^w_{t+1} = Y^w_{t+1}(s')$. Otherwise, AD prices will reflect aggregate shocks: prices will be high in states with low aggregate income.

- World rate of interest:
$$1+r = \frac{1}{\sum_s q(s)} = \frac{(Y^w_t)^{-\rho}}{\beta \sum_{s=1}^{S} \pi(s) \left[Y^w_{t+1}(s)\right]^{-\rho}} \tag{3.1}$$

- Intuition? higher world output level today lowers the real interest rate. We can also rewrite as:
$$(1+r)\,\beta \frac{E_t\left(\left(Y^w_{t+1}(s)\right)^{-\rho}\right)}{(Y^w_t)^{-\rho}} = 1$$
that is, the usual Euler equation holds at the world level.

- Consumption levels.
$$\frac{q(s)}{q(s')} = \frac{\pi(s)\, u'(C_{t+1}(s))}{\pi(s')\, u'(C_{t+1}(s'))} = \frac{\pi(s)\, u'\left(C^*_{t+1}(s)\right)}{\pi(s')\, u'\left(C^*_{t+1}(s')\right)}$$

Again the basic idea is that all country's MRS in consumption are equal. With CRRA utility, the Euler equations for both countries can be written as:

$$\frac{\pi(2)}{\pi(1)} \frac{q(1)}{q(2)} = \frac{C_{t+1}(1)^{-\rho}}{C_{t+1}(2)^{-\rho}} = \frac{C^*_{t+1}(1)^{-\rho}}{C^*_{t+1}(2)^{-\rho}}$$

From the market equilibrium conditions, we have:

$$\frac{q(1)}{q(2)} = \frac{Y^W_{t+1}(1)^{-\rho}}{Y^W_{t+1}(2)^{-\rho}} \frac{\pi(1)}{\pi(2)}$$

which implies that
$$\frac{C_{t+1}(1)}{C_{t+1}(2)} = \frac{C^*_{t+1}(1)}{C^*_{t+1}(2)} = \frac{Y^W_{t+1}(1)}{Y^W_{t+1}(2)}$$

- Since the Euler can also be expressed as:
$$C_{t+1}(s) = \left[\frac{\pi(s)\,\beta}{q(s)}\right]^{\frac{1}{\rho}} C_t$$
and
$$C^*_{t+1}(s) = \left[\frac{\pi(s)\,\beta}{q(s)}\right]^{\frac{1}{\rho}} C^*_t$$
we have:
$$Y^W_{t+1}(s) = \left[\frac{\pi(s)\,\beta}{q(s)}\right]^{\frac{1}{\rho}} Y^W_t$$
and we get the following:
$$\frac{C_{t+1}(s)}{C_t} = \frac{C^*_{t+1}(s)}{C^*_t} = \frac{Y^W_{t+1}(s)}{Y^W_t}$$

This says that *consumption growth rates are the same across countries in every state, and are equal to the growth rate of world output.*

So this is powerful test. If there is efficient risk sharing, *per capita consumption growth rates should be correlated even if national output growth rates are not.* More generally consumption co-movements should be higher across countries than output co-movements.

- Another way to look at the same thing:

$$\frac{C_{t+1}(s)}{Y_{t+1}^{w}(s)} = \mu = \frac{C_t}{Y_t^{w}(s)}$$

$$\frac{C_{t+1}^{*}(s)}{Y_{t+1}^{w}(s)} = 1-\mu = \frac{C_t^{*}}{Y_t^{w}(s)}$$

the domestic country consumes a constant fraction of world output.

This constant fraction is what we would also obtain if we solved the planner problem:

$$\max\ \mu\, U + (1-\mu)\ U^{*}$$

This is a manifestation of the first welfare theorem.

- This also provides a rationale for representative agent models: summing across agent's euler equation, we obtain

$$\bar{C}_{t+1}(s) = \left[\frac{\pi(s)\,\beta}{q(s)}\right]^{\frac{1}{\rho}} \bar{C}_t$$

so that the euler equation holds with average or total consumption.

- Notice that neither country has constant consumption across states. However, each country's consumption is internationally diversified in the sense that any consumption risk it faces is entirely due to uncertainty in global output, or in systematic output uncertainty.

3.1.3.1 Conclusion:

Simple models with one good and complete asset market imply that MRS in consumption should be equalized across countries. Under the additional assumption of isoelastic utility, consumption growth rates should be equal across countries (with a CRRA utility the consumption shares are constant). More generally consumption co-movements should be higher across countries than output co-movements.

3.1.3.2 Empirical evidence on International Risk Sharing (Obstfeld 1994 in Leiderman and Razin)

Empirical literature follows two distinct routes.

3.1.3.2.1 correlation patterns

- This approach observes that the model predicts:
  1. perfect correlation of consumption growth:

     $$\frac{C_{t+1}(s)}{C_t} = \frac{C_{t+1}^{*}(s)}{C_t^{*}} = \frac{Y_{t+1}^{W}(s)}{Y_t^{W}}$$

     while output growth correlations may differ from one.
  2. perfect correlation in levels since

     $$C_{t+1}(s) = \mu\ Y_{t+1}^{w}(s) = \frac{\mu}{\mu^{*}} C_{t+1}^{*}(s)$$

- This literature typically looks at the correlation pattern for consumption (levels or growth) and output (level or growth) and concludes that output tends to be more correlated across countries than consumption (Backus Kehoe and Kydland, JPE 1992, Stockman and Tesar (1995 AER)). Some numbers are reproduced in Table 3.1 and 3.2.[2]

[2] From Summers Heston 6.0. See program sharing.do.

TABLE 2

INTERNATIONAL COMOVEMENTS

	Contemporaneous Cross Correlations				
	With Same U.S. Variable		Within Each Country		
Country	Output	Consumption	Saving Rate with Investment Rate	Net Exports/Output with Output	Standard Deviation (%): Net Exports/Output
Australia	.25	.13	−.07	−.11	1.37
Austria	.31	.07	.29	−.42	1.12
Canada	.77	.65	.06	−.29	.79
Finland	.02	−.01	.09	−.36	1.96
France	.22	−.18	−.04	−.17	.83
Germany	.42	.39	.42	−.27	.85
Italy	.39	.25	.06	−.62	1.41
Japan	.39	.30	.50	−.03	.89
South Africa	−.15	−.23	−.60	−.56	3.35
Switzerland	.27	.25	.38	−.66	1.47
United Kingdom	.48	.43	.07	−.21	1.10
United States	1.00	1.00	.68	−.36	.42
Europe	.70	.46	...	...	...

SOURCE.—IFS. For details, see the Appendix.

NOTE.—Statistics are based on Hodrick-Prescott (1980) filtered data. Output and consumption are in logarithms. Sample period for Australia is 1960:1–1989:4; Austria, 1964:1–1990:1; Canada, 1960:1–1989:3; Finland, 1970:1–1988:2; France, 1965:1–1989:4; Germany, 1960:1–1989:4; Italy, 1970:1–1987:3; Japan, 1965:1–1990:1; South Africa, 1960:1–1989:4; Switzerland, 1967:1–1986:4; United Kingdom, 1960:1–1990:1; United States, 1960:1–1990:2; and Europe, 1970:1–1986:4. Correlations are computed for observations available for both series.

Figure 1 Backus Kehoe and Kydland JPE 1992

	Canada	France	Germany	Italy	Japan	UK	US
ROW	0.19	0.14	0.55	-0.07	-0.34	0.33	0.26
Canada		0.12	0.34	0.10	-0.01	0.45	0.75
France			0.12	0.60	0.45	-0.16	0.08
Germany				0.48	-0.10	0.24	0.68
Italy					0.58	-0.10	0.11
Japan						-0.11	0.07
UK							0.56

Table 3.1 Consumption Growth Correlation. Source: Summers Heston 6.0

	Canada	France	Germany	Italy	Japan	UK	US
ROW	0.05	0.13	0.28	0.05	-0.01	0.29	0.22
Canada		0.32	0.38	0.19	0.12	0.34	0.74
France			0.95	0.70	0.64	0.36	0.24
Germany				0.95	-0.20	0.48	0.39
Italy					0.65	0.34	0.22
Japan						0.24	0.21
UK							0.54

Table 3.2 Output Growth Correlation. Source: Summers Heston 6.0

3.1.3.2.2 Regression tests

- write preferences as:
$$u(c,\theta) = e^{\theta} \; \frac{c^{1-\rho}}{1-\rho}$$
where θ is a a vector of country characteristics (preference shifters such as population growth..)

- under complete markets,
$$\frac{\beta_n^t u_c(c_t^n, \theta_t^n)}{u_c(c_0^n, \theta_0^n)} = \frac{\beta_m^t u_c(c_t^m, \theta_t^m)}{u_c(c_0^m, \theta_0^m)}$$
so that, taking logs and assuming that $\theta_0^i = 0$,
$$\ln c_t^n = \ln c_t^m + \ln(c_0^n / c_0^m) + \ln(\beta_n/\beta_m)\,\frac{t}{\rho} + \frac{1}{\rho}(\theta_t^n - \theta_0^m)$$
so that $\ln c$ should move by equal amounts, if there are no taste shocks. In particular, no date-t variable that is uncorrelated with θ_t^n should help predict $\ln c_t^n - \ln c_t^m$

- we can also aggregate across countries: if we define $\ln c_t^w$ as the weighted sum of log country consumption (population weighted), then we can check that aggregate consumption satisfies the following euler equation:
$$t \ln \beta_n + \theta_t^n - \rho \ln c_t^n = \ln \mu_t - \rho \ln c_0^n$$
where μ_t is the common marginal rate of intertemporal substitution (it is a time effect). Summing over countries, we can write:
$$t \ln \beta^w + \theta_t^w - \rho \ln c_t^w = \ln \mu_t - \rho \ln c_0^w$$
so that, taking differences:
$$\ln c_t^n = \ln c_t^w + \ln c_0^n / c_0^w + \frac{t}{\rho} \ln \beta_n / \beta^w + \frac{1}{\rho}(\theta_t^n - \theta_t^w)$$
note that c_t^w is not weighted world consumption. If we use world consumption, we have a more complex formula! See Obstfeld 1994. Implies that consumption moves with world consumption.

- problems with pairwise correlations:
  - shocks to preferences: endogenous regressor c_t^m correlated with θ_t^m so likely to bias results downwards. Better to use world consumption.

    Example: (for exponential utility) suppose that we study an endowment economy, and assume β is constant, $c_0^n = c_0^m$, and $\rho = 1$. Then, we should have:
$$\begin{aligned} c_t^n &= c_t^m + \theta_t^n - \theta_t^m \\ c_t^n &= y_t^w + \theta_t^n - \theta_t^w \end{aligned}$$
    where $\theta_t^w = 1/N \, \sum \theta_t^n$ and the second line uses $c_t^w = y_t^w$. Assume that θ is iid and independent of y_t^w. Then, estimating the first equation gives a coefficient of:
$$\begin{aligned} var(\theta_t^n - \theta_t^w) &= var\left(\left(1 - \frac{1}{N}\right)\theta_t^n - \frac{1}{N}\sum_{m \neq n} \theta_t^m\right) \\ &= \left(1 - \frac{1}{N}\right)^2 \sigma_\theta^2 + \frac{N-1}{N^2}\sigma_\theta^2 \\ &= \sigma_\theta^2 \left[1 - \frac{1}{N}\right] \end{aligned}$$

Table 2.7. ***Regressions of national on rest-of-world consumption growth rates, 1951–72 and 1973–88***

A. 1951–72

	Canada	France	Germany	Italy	Japan	UK	US
$\Delta \log C_w$	**1.13**	0.41	− 0.27*	− 0.04*	0.16	0.67	0.64
	(0.41)	(0.32)	(0.53)	(0.43)	(0.59)	(0.50)	(0.54)
$\bar{R}^2$	0.15	0.02	− 0.04	− 0.05	− 0.05	0.04	0.02
Lags	1	0	0	0	0	0	0

B. 1973–88

	Canada	France	Germany	Italy	Japan	UK	US
$\Delta \log C_w$	0.38	**0.57**	**1.08**	0.50	**1.26**	**1.60**	0.63
	(0.71)	(0.26)	(0.35)	(0.48)	(0.43)	(0.62)	(0.40)
$\bar{R}^2$	− 0.06	0.20	0.48	0.01	0.34	0.30	0.02
Lags	1	0	1	0	0	1	2

Note: Standard errors appear below estimates of the coefficient of rest-of-world consumption growth. **Boldface entries** of this estimate are those differing from 0 at the 5% significance level or below. An asterisk (*) marks coefficients that differ from 1 at the 5% level or below. 'Lags' shows the moving-average order assumed for the equation disturbance in calculating standard errors.

Figure 2 Obstfeld 1994

$$
\begin{aligned}
\hat{\gamma} &= \frac{cov\left(c_t^m, c_t^n\right)}{var\left(c_t^n\right)} = \frac{cov\left(y_t^w + \theta_t^n - \theta_t^w, y_t^w + \theta_t^m - \theta_t^w\right)}{\sigma_c^2} \\
&= \frac{\sigma_y^2 - \sigma_\theta^2 / N}{\sigma_c^2} \\
&= 1 - \frac{\sigma_\theta^2}{\sigma_c^2}
\end{aligned}
$$

On the other hand, estimating the second equation yields:

$$\hat{\gamma}^w = 1$$

- the specification in levels includes fixed effects (period 0 terms). More generally, if we use both time series and cross section, there is the danger that consumption is near integrated, so need to difference to avoid spurious regression.

- choose specification:

$$\Delta \ln c_t^n = \delta + \gamma_w \Delta \ln c_t^w + \epsilon_t^n$$

  The results indicate that the coefficients have increased, although they remain far from 1 for many countries

- Problems with the results:

  - better to have tests of restrictions of the theory: variables known at t that should not affect consumption growth under complete markets (oil shocks or output growth)

3.1.4 Complete markets, Infinite Horizon

The present model does not allow for a complete characterization of the current account in response to unexpected shocks. To do so, consider an infinite horizon intertemporal model. For simplicity consider an endowment economy with a stochastic income that is a function of the underlying state of the world s. For simplicity, assume s is i.i.d. and takes values between 1 and S with probabilities $\pi(s)$.

3.1.4.1 T-periods, S-states complete markets household problem

Let's consider first a T-periods complete markets exchange economy model. In each period, the consumer must decide how much to consume and how much to save for next period. The environment is uncertain, and there are S states of nature in each period. For simplicity, I assume that the realizations of the states of nature are independent and state $1 \leq s \leq S$ has probability π_s.[3] The preferences of the representative household are Von-Neuman Morgenstern, represented by:

$$u(C_0) + E\left[\sum_{t=1}^{T} \beta^t u(C_t)\right] = u(C_0) + \sum_{t,s}^{TS} \beta^t \pi_s u(C_{st}) \tag{3.2}$$

where C_{st} represents consumption in state s in period t and β is the discount factor. Note that the expectations are formed according to the underlying true probabilities of the realizations of the states (rational expectations).

The household receives an exogenous and stochastic income Y_{st} in state s and period t.

In this complete market set-up, assume that there exists a complete set of Arrow-Debreu securities as of time 0, i.e. securities that pays 1 in state s and period t and 0 otherwise that people can buy and sell freely on competitive markets. Let denote q_{st} the price of one such A-D security. The household can then make contingent plans by purchasing consumption in period t, state s, C_{st} and selling contingent output Y_{st} at price p_{st}. The budget constraint for the representative household is then:

$$C_0 + \sum_{t,s}^{T,S} q_{st} C_{st} \leq Y_0 + \sum_{t,s}^{T,S} q_{st} Y_{st} \tag{3.3}$$

Note that (3.3) implies that there is only 1 budget constraint for the household problem since all contigent decisions are made as of date 0.

Maximizing (3.2) subject to (3.3) gives the following first order conditions:

$$\begin{aligned} u'(C_0) &= \mu \\ \beta^t \pi_s u'(C_{st}) &= \mu\, q_{st} \quad \forall s, \forall t \end{aligned}$$

where μ is the Lagrangian associated with the budget constraint. Defining the marginal utility of wealth as $\lambda_{st} = u'(C_{st})$ and $\lambda_0 = u'(C_0)$, we obtain:

$$\begin{aligned} q_{st} &= \beta^t \frac{\lambda_{st}}{\lambda_0} \pi_s \\ &= E\left[\beta^t \frac{\lambda_{s't}}{\lambda_0} 1_{t\{s'=s\}}\right] \end{aligned} \tag{3.4}$$

where $1_{t\{s'=s\}}$ is the indicator function that takes the value 1 if the state is s in period t and 0 otherwise. Equation (3.4) indicates that the price of the Arrow-Debreu security for state s and period t is the present discounted value of the payoff in period t (i.e. 1 if the state is s and 0 otherwise) where the discount rate is equal to the marginal rate of substitution between period 0 and period t, i.e. $\beta^t \frac{\lambda_{st}}{\lambda_0}$. That discount rate is sometimes called a *pricing kernel*, since it defines the relative weights that should be allocated to the payoffs in the different states of the world.

To gain some insight into this equation, let's take a particular example. Suppose that we are looking at the price of a security that pays 1 is state s in period 1. Then, according to (3.4), we have:

$$q_{s1} = E\left[\beta \frac{\lambda_{s'1}}{\lambda_0} 1_{t\{s'=s\}}\right] \tag{3.5}$$

and the pricing kernel from period 0 to period 1 is equal to $\beta \frac{\lambda_{s'1}}{\lambda_0}$. What is (3.5) telling us? The price of a unit of the consumption good in state s next period will be high if the marginal utility of consumption in that state λ_{s1} is itself high compared to the marginal utility of consumption today λ_0. In turn, the marginal utility of consumption is inversely related to the level of consumption in that state and period.[4] In other

[3] Alternatively, one could assume that the states follow a Markov process where the probability of state s in period t depends only on the state s' realized in period $t-1$ and is written $P[s|s']$. The algebra is slightly more cumbersome but the results and insights are unchanged.

[4] Remember that u is concave so that $u' < 0$.

words, marginal utility is high when consumption is low, so that every extra unit of consumption is very valuable. To sum up, the consumer will be willing to pay a high price for consumption goods delivered in state s period 1, when consumption is expected to be low in that state. Conversely, the state s period 1 Arrow-Debreu security has a low return when it is highly positively correlated with the pricing kernel.[5] This illustrates that asset returns are driven by how well they allow the representative household to insure consumption flows.[6]

This pricing kernel is extremely useful to price *any* asset as of time 0. Suppose that we consider an asset with price p_0 in period 0 and that pays dividend d_{s1} in period 1 with a price of p_{s1}. That asset is equivalent to a portfolio with $d_{s1}+p_{s1}$ units of the Arrow Debreu security for state s period 1. By arbitrage, the two portfolios must have the same price and therefore:

$$\begin{aligned} p_0 &= \sum_{s=1}^{S} q_{s1}\left(d_{s1}+p_{s1}\right) \\ &= \sum_{s=1}^{S} \beta\frac{\lambda_{s1}}{\lambda_0}\pi_s\left(d_{s1}+p_{s1}\right) \\ &= E\left[\beta\frac{\lambda_{s1}}{\lambda_0}\left(d_{s1}+p_{s1}\right)\right] \end{aligned}$$

Therefore the price of *any* assets today is equal to the discounted value of the future payoffs, using the pricing kernel as a discount factor. It is easy to check that this formula generalizes to any period and any state:

$$p_{st} = E\left[\beta\frac{\lambda_{s',t+1}}{\lambda_{s,t}}\left(d_{s',t+1}+p_{s',t+1}\right)\right] \tag{3.6}$$

For instance, suppose we are interested in the price of a one-period bond that pays 1 in period $t+1$ regardless of the state of nature. Denote $p_{t,t+1}$ the price of this bond in terms of period-t consumption. Then, one has:

$$p_{t,t+1} = E\left[\beta\frac{\lambda_{s',t+1}}{\lambda_{s,t}}\right]$$

One can also define the *gross* real interest rate on this one period bond as $\frac{1}{p_{t,t+1}} = r_{t,t+1}$. Then one obtains:

$$1 = E\left[\beta\frac{\lambda_{s',t+1}}{\lambda_{s,t}}r_{t,t+1}\right] \tag{3.7}$$

Let's define the pricing kernel between period t and period $t+1$ as:

$$R_{t,t+1} = \beta\frac{\lambda_{s't+1}}{\lambda_{s,t}} \tag{3.8}$$

By recursivity on equation (3.8) it is immediate that the pricing kernel between period t and period $t'>t$ is equal to:

$$\begin{aligned} R_{t,t'} &= R_{t,t+1}R_{t+1,t+2}....R_{t'-1,t'} \\ &= \beta^{t'-t}\frac{\lambda_{s't'}}{\lambda_{s,t}} \end{aligned}$$

[5] The return on the security is $E\left[\frac{1_{\{s'=s\}}}{p_{s1}}\right]$ and we can derive from (3.5) that: $E\left[\frac{1_{\{s'=s\}}}{p_{s1}}\right] = \left(1-cov\left(\frac{1_{\{s'=s\}}}{p_{s1}},\beta\frac{\lambda_{s',1}}{\lambda_0}\right)\right)/E\left[\beta\frac{\lambda_{s',1}}{\lambda_0}\right]$. The return is high the lower is the correlation with the marginal rate of substitution.

[6] In the context of an exchange economy, where the endowment at period t is non-storable, we have the following market clearing conditions: $C_{st}=Y_{st}$. This determines completely the Arrow-Debreu prices according to: $p_{st}=\beta^t\frac{u'(Y_{st})}{u'(Y_0)}\pi_s$.

Now, we can use the pricing kernel to rewrite the Household problem slightly differently. From the definition of the pricing kernel, we have: $q_{st} = R_{0,t}\pi_s$ and we can rewrite the budget constraint as:

$$C_0 + \sum_{t=1}^{T} E\left\{R_{0,t}C_t\right\} \leq Y_0 + \sum_{t=1}^{T} E\left\{R_{0,t}Y_t\right\} \tag{3.9}$$

In other words, the value today of a state contingent consumption plan $\{C_{st}\}_{s=1}^{S}$ or income plan $\{Y_{st}\}_{s1}^{S}$ in period t is equal to its expected discounted value using the equilibrium pricing kernel, $E\left\{R_{0,t}C_t\right\}$ or $E\left\{R_{0,t}Y_t\right\}$.

3.1.4.2 Infinite Horizon Stochastic Growth Model

We have now laid out the basis for the infinite-horizon complete markets problem. We use our insights to look at the more general problem of a representative households, infinite horizon model with endogenous labor supply. We assume as before that the representative household maximizes Von-Neuman Morgenstern preferences over consumption and effort that are additively separable:

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t u\left(C_t\right)\right\} \tag{3.10}$$

Under complete markets, we know from the previous section that contingent consumption streams in period t are valued with the pricing kernel $R_{0,t,}$ so that the budget constraint is:

$$\sum_{t=0}^{\infty} E_0\left\{R_{0,t}C_t\right\} \leq \sum_{t=0}^{\infty} E_0\left\{R_{0,t}Y_t\right\} \tag{3.11}$$

The first order conditions for this problem are:

$$\begin{aligned} u_c\left(C_t\right) &= \lambda_t \qquad \text{each date, each state } t \\ \beta\frac{\lambda_{t+1}}{\lambda_t} &= R_{t,t+1} \qquad \text{each date, each state } t+1 \\ & \quad \text{exhaustion of the intertemporal budget constraint (3.11)} \end{aligned} \tag{3.12}$$

3.1.4.3 The Sequential Market Set-Up

It might seem rather implausible to assume that all the markets are open as of period 0 and that all choices are made once and for all. This needs not be the case, however. It is possible to show that the exact same allocation would result in a set-up in which spot markets for consumption and labor are open every period, and one-period ahead contingent loan market exists. In practice, the sequence of spot and loan markets reproduces the set of Arrow Debreu securities (spanning), so that the preceding results hold.[7]

Let's write the Intertemporal budget constraint (3.11) in period $t+1$:

$$\sum_{s=t+1}^{\infty} E_{t+1}\left\{R_{t+1,s}C_s\right\} \leq \sum_{s=t+1}^{\infty} E_{t+1}\left\{R_{t+1,s}Y_s\right\} + a_{t+1}$$

where a_{t+1} is financial wealth at the beginning of period $t+1$. Using the Law of iterated expectations and (3.11) as of time t, one then obtains:

$$C_t + E\left\{R_{t,t+1}a_{t+1}\right\} \leq Y_t + a_t \tag{3.13}$$

Note that this budget constraint embodies the complete markets hypothesis in the following sense: at time t, the representative household knows its income Y_t and financial wealth a_t and decides how much to

[7] For a more general result along those lines, see Radner (1972) "Equilibrium of Plans, Prices and Prices Expectations in a Sequence of Markets," Econometrica.

consume today C_t and how much to save *in the different states of nature.* a_{t+1} represents purchasing power that is transferred from period t to period $t+1$.

To maximize (3.10) subject to (3.13), we use the tools of dynamic programming. The state variable is the wealth level a_t. We write $V(a_t)$ as the maximum value of (3.10). The value function satisfies the Bellman equation:[8]

$$V(a_t) = \max_{C_t, a_{t+1}} \{u(C_t) + \beta E[V(a_{t+1})]\}$$

subject to (3.13). The first-order and envelope conditions for this program are (denoting λ_t the Lagrange multiplier):

$$\begin{aligned} u_C &= \lambda_t \\ \beta V'(a_{t+1}) &= \lambda_t R_{t,t+1} \quad \text{in each state at } t+1 \\ V'(a_t) &= \lambda_t \end{aligned} \tag{3.14}$$

These conditions are identical to the conditions of the previous problem in (3.12). The third condition is identical to condition (3.8) that derives the pricing kernel, while the fourth condition identifies λ_t as the marginal value of wealth for the representative household in period t.

3.1.4.4 The current account in the infinite horizon complete markets set-up

let's consider the case where preferences are CRRA. Then the first order conditions of the consumer problem become:

$$\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho} = R_{t,t+1} \qquad \text{each date, each state } t+1$$

Since the pricing kernel is common to all agents, we have:

$$R_{t,t+1} = \beta\left(\frac{Y_{t+1}^w}{Y_t^w}\right)^{-\rho}$$

and, following similar steps to the two period case,

$$C_t = \mu\, Y_t^w$$

consumption is a constant fraction of world output, where μ equals the domestic country's share of total wealth in period 0, that is

$$\begin{aligned} \mu &= \frac{\sum_{t=0}^{\infty} E\{R_{0,t} Y_t\}}{\sum_{t=0}^{\infty} E\{R_{0,t} Y_t^w\}} \\ &= \frac{\sum_{t=0}^{\infty} E\left\{\beta Y_t^{w(-\rho)} Y_t\right\}}{\sum_{t=0}^{\infty} E\left\{\beta Y_t^{w(1-\rho)}\right\}} \end{aligned}$$

Now, in that world, the domestic country's GNP after period 0 is equal to $\mu\, Y_t^w$ and therefore, there is no saving, hence, no current account deficit or suplus. At period 0, the country's current account is simply:

$$CA_0 = Y_0 - \mu\, Y_0^w$$

3.1.4.5 Current account and market completeness

The previous result says that the implications of market structure can be important for the current account (we already know that from Kraay and Ventura or Mercereau).

- Consider first the response to a future positive domestic transitory shock:

[8] Note that we directly assumed that the value function is time-invariant. This is legitimate since the problem is time-invariant.

- in the complete market model, there is no current account surplus or deficit: output is up, but income (GNP) increases less since part of the increase in domestic output is paid out to foreigners. Consumption is up, by the same amount as income.
- in the bonds only economy, there is a current account surplus. Consumption is permanently higher as a result of the increased domestic wealth.
- The world real interest rate drops in both environments
- the bond only economy is rather efficient at sharing risk that arises from transitory shocks. Drawback is that countries drift apart permanently (random walk in the cross section: Deaton and Paxson). Wealth redistribution has permanent effects.

- consider now a permanent shock to income
  - with complete markets, consumption increases in both countries, and there is no current account deficit or surplus
  - in a bond only economy, there is no current account surplus or deficit since domestic consumption increases fully with income. While the CA response is the same, domestic and foreign consumption respond differently
  - the world interest rate is unchanged.
  - bond only economy is very inefficient at sharing risk: current account shuts down. Not a sign of efficiency.

Conclusion: cannot simply look at the current account movements. Must have some idea about the nature of the shocks and the structure of the markets.

3.1.5 International portfolio diversification

It is possible to show that a model with incomplete asset markets where only a riskless bond and risky claims to countries' uncertain future outputs are traded can replicate the results of the Arrrow-Debreu world which we have just studied.

This result extends the Lucas (1982) model, in that it does not rely on perfect pooling.

Perfect pooling imposes that there is in fact only one agent so that we cannot look at the effect of idiosyncratic shocks.

Two dates 1 and 2, N countries, S states of nature on date 2.

V_1^n is the date 1 market value of country n's uncertain date 2 output (mutual fund).

There is a riskless rate of interest r.

$$Y_1^n + V_1^n = C_1^n + B_2^n + \sum_{m=1}^{N} x_m^n V_1^m$$

$$C_2^n(s) = (1+r) B_2^n + \sum_{m=1}^{N} x_m^n Y_2^m(s)$$

FOC (express utility as a function of B_2^n):

$$u'(C_1^n) = (1+r)\beta E_1(u'(C_2^n))$$

Maximizing with respect to x_m^n gives:

$$V_1^m u'(C_1^n) = \beta E_1(u'(C_2^n) Y_2^m)$$

3.1.5.1 CRRA preferences (direct verification as in O-R)

We guess the equilibrium allocations and find equilibrium portfolios and prices which support it. The conjecture is that the equilibrium allocation is pareto efficient and therefore take the same form as in the complete asset market case.

Define μ^n as the share of initial world wealth of country n:

$$\mu^n = \frac{Y_1^n + V_1^n}{\sum_{m=1}^{N}\left(Y_1^m + V_1^m\right)}$$

We guess that a country's share of world consumption in each period and state is also μ^n.

$$C_1^n = \mu^n Y_1^W$$

$$C_2^n\left(s\right) = \mu^n Y_2^W\left(s\right)$$

This is consistent with date 2 budget constraints if the representative household holds μ^n percent of a global mutual funds which encompasses all countries second period outputs:

$$x_m^n = \mu^n$$

and if $B_2^n = 0$.

So these consumption and portfolio demands are feasible. We need to show that they are optimal (the two FOCs hold) and that date 1 budget constraint is also verified.

FOC 1 gives a unique expression for the interest rate:

$$1 + r = \frac{\left(Y_1^w\right)^{-\rho}}{\beta\sum_{s=1}^{S}\pi\left(s\right)Y_2^w\left(s\right)^{-\rho}}$$

FOC2 gives a unique expression for the share prices:

$$V_1^m = \beta E_1\left(\left(\frac{Y_2^w}{Y_1^w}\right)^{-\rho} Y_2^m\right)$$

Remark 4 Interpretation: under the proposed consumption rule, the pricing kernel for asset prices is simply the marginal rate of susbtitution evaluated at world income: $\beta\left(\frac{Y_2^w}{Y_1^w}\right)^{-\rho}$.

And it is then easy to check that for each country, date 1 budget constraint holds: resources of each country are $Y_1^n + V_1^n$, which makes a fraction μ^n of world wealth so each of them can consume a fraction μ^n of world output and purchase a fraction μ^n of mutual funds (with $B_2^n = 0$).

Conclusion: Even when the set of assets traded is limited, the equilibrium allocation is efficient. It is identical to the equilibrium reached with complete markets. This specific result depends on the utility function.

3.1.5.2 Direct Proof:

- Let's start by defining financial wealth in period 1 as $W_1^n = V_1^n$ and financial wealth in period 2 as $W_2^n\left(s\right) = R_f B_2 + \sum_m x_m^n Y_2^m$. The budget constraint can be written as:

  $$W_2^n = \bar{R}_2^n\left(W_1^n + Y_1^n - C_1^n\right)$$

  where R_2^n is the return on portfolio for country n :

  $$\begin{aligned}\bar{R}_2^n &= \frac{R_f B_2^n + \sum_m x_m^n Y_2^m}{B_2^n + \sum_m x_m^n V_1^m} = R_f\omega_f + \sum_m \omega_m^n R_2^m \\ &= R_f + \sum_m \omega_m^n\left(R_2^m - R_f\right)\end{aligned}$$

  where $R_2^m = Y_2^m / V_1^m$ and $\omega_m^n = x_m^n V_1^m / (B_2^n + \sum_m x_m^n V_1^m)$ is the share invested in country m mutual fund.

- Household in country n maximizes over C_1^n and ω_m^n :

$$U\left(W_1^n+Y_1^n\right)=u\left(C_1\right)+\beta E\left[u\left(C_2\right)\right]$$

The first order condition for consumption is

$$\left(C_1^n\right)^{-\rho}=\beta E\left[\bar{R}_2^n\left(C_2^n\right)^{-\rho}\right]$$

The FOC for asset allocation is: for all risky assets m

$$R_f E\left[\left(C_2^n\right)^{-\rho}\right]=E\left[R_2^m\left(C_2^n\right)^{-\rho}\right]$$

- Let's look for a solution where:

$$U\left(W_1^n+Y_1^n\right)=\mu+\frac{\delta^n}{1-\rho}\left(W_1^n+Y_1^n\right)^{1-\rho}$$

In that case, the envelope theorem tells us:

$$\left(C_1^n\right)^{-\rho}=\delta^n\left(W_1^n+Y_1^n\right)^{-\rho}$$

so that we can solve for

$$C_1^n=\delta^{n-1/\rho}\left(W_1^n+Y_1^n\right)$$

- We now need to check that this is indeed the solution by solving for δ^n. Under the proposed rule, we have:

$$\begin{aligned}C_2^n &= W_2^n=\bar{R}_2^n\left(W_1^n+Y_1^n-C_1^n\right)\\ &= \bar{R}_2^n\left(W_1^n+Y_1^n\right)\left(1-\delta^{n-1/\rho}\right)\end{aligned}$$

so that the FOC for consumption becomes:

$$\delta^n=\beta E\left[\left(\bar{R}_2^n\right)^{1-\rho}\right]\left(1-\delta^{n-1/\rho}\right)^{-\rho}$$

which we can solve for δ^n :

$$\delta^n=\frac{\beta E\left[\left(\bar{R}_2^n\right)^{1-\rho}\right]}{\left(1+\beta^{-1/\rho}E\left[\left(\bar{R}_2^n\right)^{1-\rho}\right]^{-1/\rho}\right)^{-\rho}}$$

Given x_m^n and the stochastic properties of R_2^n, this defines the solution.

- The FOC for asset allocation becomes:

$$R_f E\left[\left(\bar{R}_2^n\right)^{-\rho}\right]=E\left[\left(\bar{R}_2^n\right)^{-\rho}R_2^m\right]$$

This is a system of m equations in m unknown. The point to note is that since all agents face the same returns, the solution to this problem is the same so that:

$$\omega_m^n=\omega_m^{n'}=\omega_m$$

and consequently, all countries face the same portfolio return:

$$\bar{R}_2^n=\bar{R}_2$$

This implies that δ^n is common across countries and equal to:

$$\delta=\frac{\beta E\left[\left(\bar{R}_2\right)^{1-\rho}\right]}{\left(1+\beta^{-1/\rho}E\left[\left(\bar{R}_2\right)^{1-\rho}\right]^{-1/\rho}\right)^{-\rho}}$$

- We now aggregate across countries:

$$\begin{aligned}\sum_m C_1^m &= \delta^{-1/\rho}\sum_m (V_1^m + Y_1^m) = \delta^{-1/\rho}(V_1^w + Y_1^w) \\ &= Y_1^w\end{aligned}$$

where the second line uses the resource constraint in period 1. From that, it follows that

$$\delta^{-1/\rho} = \frac{Y_1^w}{Y_1^w + V_1^w}$$

and we can rewrite consumption in the first period as:

$$\begin{aligned}C_1^n &= \frac{Y_1^w}{Y_1^w + V_1^w}(Y_1^n + V_1^n) \\ &= \frac{Y_1^n + V_1^n}{Y_1^w + V_1^w}Y_1^w \\ &= \mu^n Y_1^w\end{aligned}$$

which is what we guessed before.

- Let's solve now for consumption in the second period:

$$\begin{aligned}C_2^n &= \bar{R}_2(V_1^n + Y_1^n)\left(1 - \frac{Y_1^w}{Y_1^w + V_1^w}\right) \\ &= \bar{R}_2(V_1^n + Y_1^n)\frac{V_1^w}{Y_1^w + V_1^w}\end{aligned}$$

summing over countries:

$$Y_2^w = \bar{R}_2\frac{V_1^w}{Y_1^w + V_1^w}(V_1^w + Y_1^w)$$

so that the return on the portfolio is simply:

$$\bar{R}_2 = \frac{Y_2^w}{V_1^w}$$

and we can rewrite consumption in period 2 as:

$$\begin{aligned}C_2^n &= \frac{Y_2^w}{V_1^w}(V_1^n + Y_1^n)\frac{V_1^w}{Y_1^w + V_1^w} \\ &= \frac{V_1^n + Y_1^n}{Y_1^w + V_1^w}Y_2^w \\ &= \mu^n . Y_2^w\end{aligned}$$

- Lastly, we need to solve for the portfolio policies. From the consumption FOC and the definition of R_2^n, we obtain:

$$V_1^m = \beta E\left[\left(\frac{Y_2^w}{Y_1^w}\right)^{-\rho} Y_2^m\right]$$

which is the same pricing formula as before. Now, we can write the portfolio return condition as:[9]

$$\left(R_f - \bar{R}_2\right)B_2^n + \sum_m x_m^n\left(Y_2^m(s) - \bar{R}_2 V_1^m\right) = 0$$

[9] To see this, observe that

$$\begin{aligned}W_2^n &= \bar{R}_2(Y_1^n + V_1^n - C_1^n) \\ &= \bar{R}_2\left(B_2^n + \sum_m x_m^n V_1^m\right) \\ &= R_f B_2^n + \sum_m x_m^n Y_2^m\end{aligned}$$

and this must hold for all n and all s. The only solution is:

$$\begin{aligned} B_2^n &= 0 \\ x_m^n &= x^n \end{aligned}$$

- We finally solve for x^n by observing that:

$$C_2^n = x^n.Y_2^w$$

so that

$$x^n = \mu^n$$

This completes the proof.

Remark 5 the risk free rate is defined by:

$$Y_1^{w-\rho} = \beta R_f E\left(Y_2^{w-\rho}\right)$$

Remark 6 we could have shortened the proof: as soon as we realize that the pricing kernel for all assets is identical under complete markets and the stock market economy, it results that all the real allocations are also identical.

Remark 7 Looking over at the proof, it is clear that the key step is to realize that the portfolio allocations are the same, regardless of wealth. This is a property of HARA preferences (which includes CRRA).[10] Thus, consumption in period 2 moves in proportion to the return on a common world return. This eliminates the gains from further consumption insurance.

Remark 8 If people differ in risk aversion, they will still select the same risky portfolio, but they will differ in the mix of riskfree and risky assets. With preferences that differ more generally, people in different countries may choose different portfolios. In that case, more assets are needed to "synthetize" the AD securities.

3.1.5.3 Empirical implications

Under the assumptions of the model, investors from all countries hold the same portfolio of risky assets (perhaps differ in mix risky/riskless if risk aversion varies across countries). Standard mutual fund theorem from Finance.

3.1.6 The International Consumption CAPM Model

We build on the framework that we have just studied.

The preceding model gives:

$$V_1^m = E_1\left[\frac{\beta u'(C_2)}{u'(C_1)} Y_2^m\right]$$

$$V_1^m = E_1\left[\frac{\beta u'(C_2)}{u'(C_1)}\right] E_1\left(Y_2^m\right) + Cov_1\left(\frac{\beta u'(C_2)}{u'(C_1)}, Y_2^m\right)$$

$$V_1^m = \frac{E_1\left(Y_2^m\right)}{R_f} + Cov_1\left(\frac{\beta u'(C_2)}{u'(C_1)}, Y_2^m\right)$$

[10] Hara preferences are of the following form:

$$u(C) = \frac{1-\rho}{\rho}\left(\frac{\delta C}{1-\rho} + \eta\right)^{\rho}$$

it covers CRRA ($\eta = 0$) and CARA ($\rho = \infty$) as special cases.

The date 1 price of uncertain output is the sum of its expected value (discounted) and its value as consumption insurance. An asset that pays off unexpetedly well in date 2 when consumption is unexpectedly low (marginal utility high) commands a high price.

Define the ex-post real rate of return to a share in country m's output,

$$R^m = \frac{Y_2^m}{V_1^m}$$

We get:

$$E_1\left(R^m\right) - R_f = -R_f \ Cov_1\left(\frac{\beta u'\left(C_2\right)}{u'\left(C_1\right)},\ R^m - R_f\right)$$

The expected premium an asset must yield over the riskless rate of return depends negatively on the covariance of the asset's excess return with the rate of growth of the marginal utility of consumption (same intuition as before). The excess return is positive if the asset return is positively correlated with world consumption growth. What matters is the hedging value of the asset. This is true even in the case of incomplete markets, as long as the first order conditions hold. Of course the equilibrium must exist, which is a non trivial concern for general models with incomplete markets.

3.1.6.1 Application: the equity premium puzzle

For a CRRA utility function

$$E_1\left(R^m\right) - R_f = -R_f \ Cov_1\left(\left(\frac{C_2}{C_1}\right)^{-\rho}, R^m - R_f\right)$$

Annual consumption growth is stable. The covariance of consumption and excess returns is far too low to explain actual excess returns (equity premium) unless the degree of risk aversion is umplausibly high. But if agents were very risk averse, then they would diversify risk much more than what we see in the data!

3.2 The Infinite Horizon consumption-based CAPM

Representative agent of country n maximises at date t:

$$U_t = E_t\left[\sum_{s=t}^{\infty} \beta^{s-t} u\left(C_s^n\right)\right]$$

The country budget constraint at date s is:

$$B_{s+1}^n + \sum_{m=1}^{N} x_{m,s+1}^n V_s^m = \left(1 + r_s\right) B_s^n + \sum_{m=1}^{N} x_{m,s}^n \left(Y_s^m + V_s^m\right) - C_s^n$$

where V_s^m is the date s market price of a claim to the entire income stream of country m in all future periods, $x_{m,s}^n$ is the share of country m's security purchased by agent n in period $s-1$ and r_s is the one period real interest rate between period $s-1$ and s. Maximizing the utility function subject to the BC with respect to $x_{m,s}^n$ gives:

$$V_s^m u'\left(C_s^n\right) = \beta E_s\left(u'\left(C_{s+1}^n\right)\left(Y_{s+1}^m + V_{s+1}^m\right)\right)$$

and the usual euler condition is

$$u'\left(C_s^n\right) = \left(1 + r_{s+1}\right) \beta E_s\left(u'\left(C_{s+1}^n\right)\right)$$

So it is easy to see that:[11]

$$\begin{aligned} V_t^m &= E_t\left[\sum_{s=t+1}^{\infty}\frac{\beta^{s-t}u'(C_s)}{u'(C_t)}Y_s^m\right] \\ &= \sum_{s=t+1}^{\infty}R_{t,s}E_t(Y_s^m)+\sum_{s=t+1}^{\infty}cov_t\left(\frac{\beta^{s-t}u'(C_s)}{u'(C_t)},Y_s^m\right)\end{aligned}$$

where $\bar{R}_{t,s}$ is the date t market discount factor for non contingent date s consumption:

$$\begin{aligned} \bar{R}_{t,s} &= E_t\left(\frac{\beta^{s-t}u'(C_s)}{u'(C_t)}\right) \\ &= E_t(R_{t,s})\end{aligned}$$

Remark 9 the value of the country mutual fund does not depend upon which country's consumption we use to evaluate future profits. While the pricing kernels may not be unique under incomplete markets, they must agree on th existing set of assets. However, agent's need not agree on the price of financial assets that are not present in the economy.

All the results of the two period model generalize immediately. In particular the result that each country's consumption is a constant fraction of world output Y_t^w.

It is possible to show that the solution to the optimization problem of the consumer described above is dynamically consistent, i.e. that it is optimal to implement at each date the program chosen at date 1. There is no need for further asset trade after date 1.

3.3 Estimating the gains from international risk sharing

Lucas (1987) estimates the welfare cost of variability in United States consumption. He comes up with a small number (1/3 % of yearly consumption for a representative consumer). So it seems logical, that if one takes the same type of framework, the gains from eliminating national idiosyncratic risk is going to be found small. And in fact, Cole and Obstfeld (1991) [JME] do come up with small numbers when estimating the gains from international risk-sharing (around 0.2% of output per year). Their approach is in the spirit of Lucas (1982) [JME] but a major difference is the existence of **terms of trade effect**.

Terms of trade adjustments provide automatic pooling of risks since they are negatively correlated with growth in the export sector (extreme case is Cobb Douglas preferences which imply unitary price elasticities).

They assume one representative agent per country (this assumes complete integration of financial markets in each country).

3.3.1 Perfect international capital market integration

Lucas (1982) solves for the pooled equilibrium (world agent who owns half of each country endowment). Cole and Obstfeld (1991) solves for the planner's equilibrium with their two representative agents. Solutions to this problem are Pareto optima whose decentralized equilibria counterparts involve generally different national wealth and consumption levels.

[11] where we iterate and use the condition that

$$\lim_{T\to\infty}\beta^{T-t}u'(c_T)/u'(c_t)V_T^m=0$$

3.3.1.1 The Lucas (1987) calculation

- Consider a representative agent with the following CRRA utility:

$$U = E_t \sum_{s=t}^{\infty} \beta^{s-t} \frac{C_s^{1-\rho}}{1-\rho}$$

- suppose that consumption fluctuates (because risks are imperfectly shared or otherwise) so that:

$$C_s = G^{s-t} \bar{C} \; e^{\epsilon_s - \frac{1}{2}\sigma_\epsilon^2}$$

  where ϵ_s is normally distributed $N\left(0, \sigma_\epsilon^2\right)$, so that $E_t\left[C_s\right] = G^{s-t}\bar{C}$

- we can calculate expected utility:

$$\begin{aligned} E_t\left(C_s^{1-\rho}\right) &= G^{(s-t)(1-\rho)} \bar{C}^{1-\rho} \; E_t\left(e^{(1-\rho)\epsilon_s - \frac{(1-\rho)}{2}\sigma_\epsilon^2}\right) \\ &= G^{(s-t)(1-\rho)} \bar{C}^{1-\rho} \; e^{\frac{(1-\rho)^2}{2}\sigma_\epsilon^2 - \frac{(1-\rho)}{2}\sigma_\epsilon^2} \\ &= G^{(s-t)(1-\rho)} \bar{C}^{1-\rho} \; e^{-\frac{\rho(1-\rho)}{2}\sigma_\epsilon^2} \end{aligned}$$

  so that

$$U_t = \frac{\bar{C}^{1-\rho}}{1-\rho} \frac{1}{1-\beta G^{1-\rho}} e^{-\frac{\rho(1-\rho)}{2}\sigma_\epsilon^2}$$

- equivalently, if consumption grows along a constant path, utility is:

$$\bar{U}_t = \frac{\bar{C}^{1-\rho}}{1-\rho} \frac{1}{1-\beta G^{1-\rho}}$$

- so we can ask, what is the 'compensating variation', i.e. the change in level consumption that compensates for the fluctuations:

$$\begin{aligned} \tau &= e^{\frac{\rho}{2}\sigma_\epsilon^2} - 1 \\ &\approx \frac{\rho}{2}\sigma_\epsilon^2 \end{aligned}$$

  with $\rho = 5$, and $\sigma_\epsilon^2 = 0.000708$, find that $\tau = 0.0035$ or about 0.35%!!

3.3.1.2 The Open Economy Planner

Home country residents maximize their consumption of home and foreign goods:

$$U_0 = E_0 \left[\sum_{t=0}^{+\infty} \beta^t u\left(x_t, y_t\right)\right]$$

There are no frictions on the good market.

The planner's programme is:

$$\max\left[\mu u\left(x, y\right) + \left(1-\mu\right) u\left(x^*, y^*\right)\right]$$

subject to the ressource constraints:

$$x + x^* = X$$

$$y + y^* = X$$

Optimal allocations are determined by the FOC and the good markets equilibria. μ is a planner weight that determines relative wealth levels in the counterpart market equilibrium.

FOC:

$$\frac{u_j\left(x, y\right)}{u_j\left(x^*, y^*\right)} = \frac{1-\mu}{\mu}$$

with $j = x, y$

So the ratio of marginal utilities must be constant across goods and states of nature. This also means that national marginal utilities from consuming any good are perfectly correlated.

Assume that

$$u(x,y) = \frac{\left(x^{\theta}y^{1-\theta}\right)^{1-\rho}}{1-\rho}$$

The planning solution is:

$$\begin{aligned}
\frac{x^{\theta-1}y^{1-\theta}}{x^{*\theta-1}y^{*1-\theta}}.\frac{\left(x^{\theta}y^{1-\theta}\right)^{-\rho}}{\left(x^{*\theta}y^{*1-\theta}\right)^{-\rho}} &= \frac{1-\mu}{\mu} \\
\frac{y^{-\theta}x^{\theta}}{y^{*-\theta}x^{*\theta}}.\frac{\left(x^{\theta}y^{1-\theta}\right)^{-\rho}}{\left(x^{*\theta}y^{*1-\theta}\right)^{-\rho}} &= \frac{1-\mu}{\mu}
\end{aligned}$$

Equating both lines:

$$\frac{y}{y^*} = \frac{x}{x^*} = \kappa$$

so that

$$\begin{aligned}
\frac{x^*}{X} &= \frac{y^*}{Y} = \frac{1}{1+\kappa} \\
\frac{x}{X} &= \frac{y}{Y} = \frac{\kappa}{1+\kappa}
\end{aligned}$$

and plugging back into the FOC:

$$\kappa = \left(\frac{1-\mu}{\mu}\right)^{-1/\rho}$$

so that:

$$\begin{aligned}
x &= \omega X \\
y &= \omega Y \\
x^* &= (1-\omega) X \\
y^* &= (1-\omega) Y
\end{aligned}$$

with

$$\omega = \frac{1}{1+\left[\frac{1-\mu}{\mu}\right]^{1/\rho}}$$

If $\mu = 1/2$ then we have the Lucas pooled equilibrium. For other values however, there are different levels of national wealth and different efficient market outcomes. In all cases, national consumptions of the 2 goods are perfectly correlated because countries insure each other agains country specific shocks.

3.3.2 Decentralized equilibrium under autarky.

Trade is balanced. Denote p the price of good Y.

For a regime of portfolio autarky, we have no theoretical reason to believe that in general allocations will be efficient. In this particular case, however, we will see that it is possible to find a planner's weight such that the financial autarky allocation and the planner's solution are the same.

Budget constraint are:

$$\begin{aligned}
x + py &= X \\
x^*/p + y^* &= Y
\end{aligned}$$

With cobb douglas preferences, expenditure shares satisfy:

$$(1-\theta)\, x = \theta p y$$

and:

$$\begin{aligned} x &= \theta X \\ y &= (1-\theta)\,X/p \\ x^* &= \theta p Y \\ y^* &= (1-\theta)\,Y \end{aligned}$$

Market clearing gives:

$$x + x^* = \theta X + \theta p Y = X$$

so:

$$p = \frac{(1-\theta)}{\theta}\frac{X}{Y}$$

So equilibrium consumptions are

$$\begin{aligned} x &= \theta X \\ y &= \theta Y \\ x^* &= (1-\theta)\,X \\ y^* &= (1-\theta)\,Y \end{aligned}$$

This is a solution to the planner's problem if:

$$\mu = \frac{1}{\left(1 + \left[\frac{(1-\theta)}{\theta}\right]^{\rho}\right)}$$

If $\theta = 0.5$, then $\mu = 0.5$ and we find back the pooling result of Lucas even without asset markets!!

In this very specific case, autarky and free trade allocations are the same. The market solution under autarky is a member of the Pareto efficient family of planning solutions.

What is going on? Terms of trade adjustment automatically pool national output risks. In that case, a country with a high output sees the terms of trade deteriorate just enough to transfer purchasing power to the foreign country and insulate domestic consumption.

Since any Pareto-optimal allocation corresponds to the competitive equilibrium of an economiy with complete, integrated asset markets, financial integration has no observable implications in these examples. This suggests that looking at the empirical evidence can be tricky.

3.3.3 Theoretical Robustness of the results

This extreme result disappears if one has:

3.3.3.1 non specialization in production

suppose both countries produce a common good z with different endowments (not perfectly correlated) and expenditure share θ_z. The budget constraint becomes:

$$\begin{aligned} p_x x + p_y y + z &= p_x X + Z \\ p_x x^* + p_y y^* + z^* &= p_y Y + Z^* \end{aligned}$$

and the demands are:

$$\frac{p_x x}{\theta_x} = \frac{z}{\theta_z} = \frac{p_y y}{\theta_y}$$

so that

$$\begin{aligned} p_x x &= \theta_x\,(p_x X + Z) \\ z &= \theta_z\,(p_x X + Z) \\ p_y y &= \theta_y\,(p_x X + Z) \end{aligned}$$

and the equilibrium on good X market implies:

$$\begin{aligned}\theta_x (p_x X + p_y Y + Z + Z^*) &= p_x X \\ \theta_y (p_x X + p_y Y + Z + Z^*) &= p_y Y\end{aligned}$$

so that (since $\theta_x + \theta_y = 1 - \theta_z$):

$$(1-\theta_z)(Z+Z^*) = \theta_z (p_x X + p_y Y)$$

and

$$\begin{aligned}p_x &= \frac{\theta_x}{\theta_z}\frac{(Z+Z^*)}{X} \\ p_y &= \frac{\theta_x}{\theta_z}\frac{(Z+Z^*)}{Y}\end{aligned}$$

and the demand for good x satisfies:

$$x = \left(\theta_x + \theta_z \frac{Z}{Z+Z^*}\right) X$$

so that the share of good X that is consumed varies with the endowment shock of good Z. In general, perfect risk sharing is not possible. A shock in a common industry (Z) is transmitted negatively: an increase in Z lowers the price of z relative to the other goods in the other country.

3.3.3.2 non tradable goods

Suppose good N is non-tradable and its expenditure share is θ_n. Then, we have that:

$$\begin{aligned}p_x x &= \theta_x (p_x X + N) \\ p_y y &= \theta_y (p_x X + N) \\ n &= \theta_n (p_x X + N)\end{aligned}$$

and similar equations for the foreign country. Equilibrium on the market for n implies:

$$n = N = \theta_n (p_x X + N)$$

so that

$$p_x = \frac{1-\theta_n}{\theta_n}\frac{N}{X}$$

and the demand for good x satisfies:

$$\begin{aligned}x &= \frac{\theta_x}{1-\theta_n} X \\ y &= \frac{\theta_x}{1-\theta_n} Y\end{aligned}$$

so that consumption of tradable is perfectly insured as before. Unlike the previous case, however, the fluctuations in nontradable imply that the marginal rates of substitution are not constant, i.e. the condition for perfect risk sharing is not satisfied:

$$\begin{aligned}\frac{u_x}{u_x^*} &= \frac{x^{\theta_x - 1} y^{\theta_y} N^{\theta_n}}{x^{*\theta_x - 1} y^{*\theta_y} N^{*\theta_n}} . \frac{\left(x^{\theta_x} y^{\theta_y} N^{\theta_n}\right)^{-\rho}}{\left(x^{*\theta_x} y^{*\theta_y} N^{*\theta_n}\right)^{-\rho}} \\ &= \left(\frac{\theta_x}{\theta_y}\right)^{-\theta_n - (1-\theta_n)\rho} \left(\frac{N}{N^*}\right)^{\theta_n (1-\rho)} .\end{aligned}$$

and it fluctuates with N/N^*

3.3.3.3 Investment

Introducing investment also destroys the results except for knife-edge cases (log preferences)

3.3.4 Empirical evaluation

Cole and Obstfeld look at departures from those specific cases and calibrate their model to US Japan data (endowment economy: exogenous output processes). They postulate a two state Markov process for each country (low growth, high growth).

$$\begin{aligned} X_{t+1} &= \left(1+\varepsilon_t^X\right) X_t \\ Y_{t+1} &= \left(1+\varepsilon_t^Y\right) Y_t \end{aligned}$$

So the state of the world economy is given by the vector $\left(\varepsilon_t^X, \varepsilon_t^Y\right)$ which can take 4 possible values.

Instead of assuming a Cobb Douglas utility (unitary terms of trade) they look at the CES/isoelastic case.

$$u(x,y) = \frac{\left[(x^\rho + y^\rho)^{\frac{1}{\rho}}\right]^{1-R}}{1-R}$$

(note that the log case corresponds to $\rho = 0$ in their notation).

They calibrate their model so that it matches mean, standard deviation and lagged autocorrelation of output growth as well as the correlation of output growth rates between the US and Japan. Calibration boils down to choosing the two possible realizations of the Markov growth processes and the transition probability (4X4 matrix).

The authors impose a perfect symmetry between the two countries. The two countries start off with $X_0 = Y_0$. An initial pair of growth rates is drawn from the steady state distribution. Consumptions are determined given the degree of financial market integration assumed . Etc...

When perfect capital market integration is assumed, since the two economies are exactly symmetric, there is perfect pooling.

They replicate the experiment 10000 times and look at the average value they obtain. They use $\beta = 0.98$, a 50 period economy and various degrees of risk aversion (2 to 30) and ρ (0.25 to 1).

It is then possible to compute the welfare loss (in terms of output) due to autarky.

$$U^A(X_0) = U^I((1-\delta) X_0)$$

δ is extremely small (at most 0.49% of GDP per year in the most extreme case). It is increasing in the degree of risk aversion (quite intuitive) and increasing in ρ. This latter result comes from th weakening of the terms of trade effect as substitution across goods increases.

Cole and Obsfeld's conclusion is that since the gains from international risk sharing are low, *even small frictions are enough to deter asset trade.* This would explain why even though asset markets seem to be integrated there is little international diversification of risk.

3.3.5 Qualifications

1. Investment is neglected

2. Additional shocks

3. Heterogeneous individuals

4. Other growth effects

Table 1

Welfare loss due to a ban on international diversification (fraction of national product per year).[a]

R	ρ = 0.25	0.50	0.75	1.00
2	0.000045 (0.000087)	0.000347 (0.000174)	0.000906 (0.000261)	0.001722 (0.000349)
4	0.000120 (0.000074)	0.000619 (0.000148)	0.001497 (0.000223)	0.002760 (0.000300)
6	0.000157 (0.000066)	0.000752 (0.000132)	0.001790 (0.000201)	0.003284 (0.000272)
8	0.000177 (0.000060)	0.000829 (0.000122)	0.001965 (0.000186)	0.003605 (0.000255)
10	0.000191 (0.000057)	0.000882 (0.000116)	0.002088 (0.000177)	0.003841 (0.000245)
30	0.000195 (0.000057)	0.000987 (0.000120)	0.002469 (0.000199)	0.004851 (0.000310)

[a]For a given CES utility function parameter ρ and risk aversion coefficient R, the reported number is the fraction by which base-year output must be reduced to yield a welfare loss equal to that caused by a ban on international asset trade. Expected utility levels are calculated as the average of utility realizations in 10,000 independent replications of a symmetric two-country world economy in which national output growth rates follow a two-state Markov process. (Approximate standard errors appear in parentheses below the output-loss estimates.)

Figure 3 Cole and Obstfeld JME 1991

3.3.6 Other estimates of the gain from risksharing

The literature is all over the place: from 0 to 100% of consumption. Why?

why some estimates are very high:

- some models match the equity premium (Lewis JPE 1996). Requires high degree of risk aversion. So not surprising that the gains are large. indicates that it is a bit difficult to calibrate these models. In general, the gains estimated from the 'finance' approach are very large. Because they implicitly assume that the equity premium is here to stay.

- some models concentrate on near integrated processes: much larger endowment uncertainty (suppose that $Y_{t+1} = \rho Y_t + \epsilon_{t+1}$ even with small σ_ϵ^2, $var\,(y_t)$ can be large (Shiller Athanasoulis (1995 NBER)

- endogenous growth (Obstfeld 1994)

why some estimates are very low:

- high risk free real interest rate.

- too much risk sharing via terms of trade: imply that consumptions are already correlated. Seems to contradict the evidence

Obstfeld 1994

Pallage and Robe (2000)

3.4 Another model of asset trade: Martin and Rey (JIE 2004)

Predecessors

- Helpman Razin [1978]

  Integration of a stock market economy in a Hecksher Ohlin model of trade.

- Svensson 1988 [AER]

  This paper develops a theory of international trade in assets based on comparative advantage.

- Acemoglu Zilibotti, 1997 [JPE]

3.4.1 The model

Model of endogenously incomplete markets. Draws a parallel between trade in assets and trade in goods. Borrows from the 'new trade theory', trade in variety type model. Previous models look at the interaction between specialization/endowment and trade in assets (Svensson 1988 and Helpman Razin 1978).

4 key assumptions:

- number of assets is endogenous.
- Fixed costs in the investment technology (so limited number of assets proposed in equilibrium)
- Assets are imperfect substitutes
- Cross-border transaction costs

Remark 10 it is unclear whether this is really a model of incomplete markets: incomplete markets refers to situations where there is residual idiosyncratic risk that is uninsured. This is not the case here. Can think of it as a model with different degrees of aggregate risk. With few projects, chances are high that output will be zero. With many projects, less so.

Remark 11 here we will also have the interesting result that imperfect competition increases idiosyncratic risk (since we hold more of the domestic asset)

3.4.1.1 Assumptions

- 2 periods, 2 countries. Countries differ in size. Unit mass of agents. N domestic agents, N^* foreign agents(represents population, not economic size here).
- Uncertainty: tomorrow's state of the world is the realization of some random variable $s \in [0, S]$. Assume that s is uniformly distributed.
- Technology:

  - Arrow Debreu assets, as in Acemoglu and Zilibotti.

    A project is an A-D security that pays y in state of the world s, 0 otherwise.

    Interpretation: returns to firms are risky. Here think about the 'strips'.

  Remark 12 Different projects are imperfect substitute (provide consumption in different states of the world).

  Remark 13 Variety improves risk sharing: taste for variety.

- We define $\mathcal{A}_i$ the set of projects on $[0, S]$ that agent i is undertaking and it's mass: $\mu(A_i) = \mu_i$.Cost of producing a mass μ of projects: $f(\mu)$ with $f(0) = 0$ and $f', f'' > 0$. Can think of monitoring becoming more costly. Each $\mathcal{A}_i$ is disjoint. It never makes sense to produce an asset that is already produced, as long as some states are not spanned.
- We also define $\mathcal{A} = \cup_i \mathcal{A}_i$ for domestic agents and $\mathcal{A}^*$ the foreign counterpart. These represent the sets of domestic and foreign projects, respectively. We denote also $\mathcal{A} \backslash \mathcal{A}_i$ the set of domestic assets not produced by agent i.

Remark 14 we will look at a situation where S is large enough that the total number of projects is smaller than the number of states, so that $\sum_i \mu_i < S$.

- Endowment in the first period: Y_1 and Y_1^* freely traded.
- Timing:
  - period 1: consume and invest in the risky projects, at home an abroad. Chooses C_1^i and μ_i.
  - period 2: consume the return on investment that depends upon the state of the world that is realized.
- Cross border transaction costs: iceberg cost τ for international financial transaction, both on the purchase and the dividend flow.

Denote $x^i(s)$ the asset demand for AD security for state s for a domestic agent i and $x^{j*}(s)$ the asset demand for some foreign agent j. Note that if $s \in \mathcal{A}$ this is a domestic asset, and $s \in \mathcal{A}^*$, this is a foreign asset.

Budget constraint in period 1:

$$C_1^i + f(\mu_i) + \int_{\mathcal{A}} p(s)\, x(s)\, ds + (1+\tau) \int_{\mathcal{A}^*} p(s)\, x(s)\, ds = Y_1 + \int_{\mathcal{A}_i} p(s)\, ds$$

- the right hand side represents current income and the revenues from selling each project s on $\mathcal{A}_i$ at price $p(s)$.
- The left hand side represents current consumption, the cost of setting up the projects and the demand for domestic and foreign shares (both domestically produced and otherwise)

Budget constraint in period 2:

$$C_2(s) = \begin{cases} x(s)\, y & \text{if} \quad s \in \mathcal{A} \\ (1-\tau)\, x(s)\, y & \text{if} \quad s \in \mathcal{A}^* \\ 0 & \text{otherwise} \end{cases}$$

Remark 15 it is possible for consumption to be zero in the second period. This is an artifact of the assumption that firms are AD securities. If we convexify a bit more, so that firms pay out a kernel over the state of nature, the results still go through as long as we cannot replicate the complete market equilibrium

- Preferences:

$$U = C_1 + \beta E \left(\frac{C_2^{1-\rho}}{1-\rho} \right)$$

  where ρ is the degree of risk aversion.

If we substitute the expression for C_2 into preferences, we obtain:

$$U = C_1 + \frac{\beta}{S} \frac{y^{1-\rho}}{1-\rho} \left[\int_{\mathcal{A}} x(s)^{1-\rho} ds + (1-\tau)^{1-\rho} \int_{\mathcal{A}^*} x(s)^{1-\rho} ds \right]$$

This looks very much like a trade in variety model with dixit-stiglitz preferences.

Remark 16 The linearity of the assumption for preferences in terms of first period consumption garanties that there is no smoothing component to consumption allocation.

3.4.2 The solution

- Comptetition is monopolistic. Each AD project is priced by a monopolist. This reflects the fact that firms can retain shares to increase share prices and extract more revenues (another parallel with the trade literature).
- Consider first the foreign demand for domestic assets. We can write the foreign budget constraint as:

$$C_1^{j*} + f\left(\mu_j^*\right) + (1+\tau)\int_{\mathcal{A}} p(s)\, x_j^*(s)\, ds + \int_{\mathcal{A}^*} p(s)\, x_j^*(s)\, ds = Y_1^* + \int_{\mathcal{A}_j^*} p(s)\, ds$$

  and solve for the foreign demand for domestic assets by maximizing over $x_j^*(s)$ for $s \in \mathcal{A}_i$:

$$\begin{aligned} p(s) &= \frac{\beta}{S}\, y^{1-\rho} \left(\frac{1-\tau}{1+\tau}\right)^{1-\rho} (1+\tau)^{-\rho}\, x_j^*(s)^{-\rho} \\ x_j^*(s)(1+\tau) &= p(s)^{-1/\rho} \left(\frac{\beta}{S}\, y^{1-\rho}\right)^{-1/\rho} \left(\frac{1-\tau}{1+\tau}\right)^{-\frac{\rho-1}{\rho}} \end{aligned}$$

  common for all j

- Similarly, we can solve for the domestic demand for asset $s \in \mathcal{A}_i$ for $i' \neq i$:

$$\begin{aligned} p(s) &= \frac{\beta}{S}\, y^{1-\rho} x_{i'}(s)^{-\rho} \\ x_{i'}(s) &= p(s)^{-1/\rho} \left(\frac{\beta}{S}\, y^{1-\rho}\right)^{-1/\rho} \end{aligned}$$

  common for all $i' \neq i$.

- In both cases, the perceived elasticity of demand is $1/\rho$, as usual with Dixit Stiglitz preferences. We need to assume that $\rho < 1$ if we want the model to be well behaved (otherwise the markup is negative and the solution is to supply 0). This is a limitation of the analysis. Could perhaps be relaxed with Epstein Zin preferences.
- To solve for the optimal supply and price for AD asset $s \in \mathcal{A}_i$, let's rewrite the domestic budget constraint using the equilibrium on the domestic asset market $\left(\sum_{i' \neq i} x_{i'}(s) + (1+\tau)\sum_j x_j^*(s) = 1 - x_i(s)\right)$ as:

$$C_1^i + f(\mu_i) + (1+\tau)\int_{\mathcal{A}^*} p(s)\, x(s)\, ds + \int_{\mathcal{A}\backslash\mathcal{A}_i} p(s)\, x(s)\, ds = Y_1 + \int_{\mathcal{A}_i} p(s)\,(1 - x(s))\, ds$$

  We now maximize U over $x(s)$ subject to the budget constraint and the foreign demand for domestic asset. We obtain:

$$\begin{aligned} p(s) &= \frac{1}{1-\rho}\frac{\beta}{S} y^{1-\rho}\, x_i(s)^{-\rho} \\ x_i(s) &= (1-\rho)^{-1/\rho}\, p(s)^{-1/\rho} \left(\frac{\beta}{S}\, y^{1-\rho}\right)^{-1/\rho} \end{aligned}$$

  this is the usual formula: $p = MC/(1 - 1/\sigma)$ where σ is the elasticity of demand. The markup is constant and equal to $1/1 - \rho$.

- Substituting the foreign demand for domestic asset and using the equilibrium condition for asset s :

$$\begin{aligned} x_i(s) &= (1-\rho)^{-1/\rho}\, x_{i'}(s) \\ &= \delta x_{i'}(s) \\ x_i(s) &= \frac{\delta}{\phi}(1+\tau)\, x_j^*(s) \end{aligned}$$

Proposition 3.4.1 (Home Equity Bias) *No full diversification: the producer of a project retains a higher share:*

$$\begin{aligned} x_i(s) &= \delta x_{i'}(s) \\ &= \frac{\delta}{\phi} x_j(s) \end{aligned}$$

This implies that projects or firms have a nationality. For domestic demands, this depends only upon imperfect competition. When $\delta = 1$, we have $x_i(s) = x_{i'}(s)$.

- Equilibrium on the asset market s implies:

$$x_i(s) + (N-1)\, x_{i'}(s) + N^* (1+\tau)\, x_j^*(s) = 1$$

$$x_i(s) = \frac{\delta}{(N-1) + \phi N^* + \delta}$$

where:

$$\begin{aligned} \phi &= \left(\frac{1-\tau}{1+\tau}\right)^{\frac{1-\rho}{\rho}} \\ \delta &= (1-\rho)^{-1/\rho} \end{aligned}$$

$\phi < 1$ measures the importance of cross border transaction costs: if $\tau = 0$, $\phi = 1$. On the other hand, if $\tau = 1$, $\phi = 0$.

$\delta > 1$ measures the important of the imperfect competition. When $\rho = 1$, $\delta = \infty$. If markets were competitive, on the other hand, we would have $\delta = 1$ (corresponds formally to $\rho = -\infty$).

Proposition 3.4.2 *There is more diversification in a large country than a small country: for $s \in \mathcal{A}_i$ and $s' \in \mathcal{A}_j$:*

$$x_i(s) < x_j^*(s')$$

We can check that this is true by noting that

$$\frac{x_i(s)}{x_j(s')} = \frac{(N^*-1) + \phi N + \delta}{(N-1) + \phi N^* + \delta} < 1$$

since $\phi < 1$. Only when $\phi = 1$ do we have $x_i(s) = x_j(s')$. Note that this result is independent of the degree of imperfect competition. It would also occur under perfectly competitive.

- We now solve for equilibrium prices:

$$p(s) = \frac{\beta}{S} y^{1-\rho} \;\; ((N-1) + \phi N^* + \delta)^{\rho}$$

We see that prices increase with market size.

Proposition 3.4.3 *Demand effect: asset prices are higher in the larger market: for $s \in \mathcal{A}$ and $s' \in \mathcal{A}^*$*

$$p(s) > p(s')$$

- we now solve for the supply of assets by each agents. Can show that the FOC is:

$$f'(\mu) = p$$

Proposition 3.4.4 *financial markets are more developed in a larger country:*

$$\mu > \mu^*$$

Home bias arises for two reasons:

- imperfect competition
- transaction costs

3.4.3 Welfare analysis:

two sources of inefficiency:

- *pecuniary externality.* Opening a new project reduces the risk that other agents face. This makes all the other projects more attractive. But the owner of the additional project does not internalize this effect. This creates a coordination failure: the more projects there are, the more I want to offer projects (since p increases). This leads to too little diversification in equilibrium
- *imperfect competition*:
  - leads agents to retain too much ownership of project;
  - monopolistic power also leads to too many projects opened

So it is unclear a priori whether there are too many or too few projects. Can show by solving the planner's problem that there is always too little diversification.

3.4.4 Empirical analysis

- Portes and Rey (2002) uses a gravity model to look at financial flows.
- Only model that works in real trade (embarassingly so).
- Gravity model have two explantory variables:
  - product of size (economic)
  - distance
- New trade theory (HK) has the first variable, not always the second. Distance proxies for transaction costs.
- Portes and Rey uses data on bilateral trade in assets between 14 OECD countries for 8 years. Variables that proxy information costs: volume of telephone calls, number of foreign banks branches.

$$\ln T_B^A = \beta_1 \ln (Mcap)_A + \beta_2 \ln (Mcap_B) + \beta_3 \ln (\text{distance}) + \beta_4 \text{ information variables}$$

- Martin Rey model predicts that both variables should be there (transaction costs)
  - interpret transaction costs as information
  - use proportional transaction costs for ϕ
  - find strong results

$$\ln T_B^A = \ln (Mcap)_A + \ln (C_B) + \ln \phi + \frac{1}{\rho} \ln (y/Np_A) + \ln \left(\beta^{1/\rho}/y\right)$$

where T_B^A represents sales of portfolio equity from A to B

TABLE 2 – BILATERAL EQUITY FLOWS 1989-96

equityij	(1)	(2)	(3)	(4)	(5)[a]	(6)[a]	(7)[e]	(8)[e]
mktcapi	0.987 (.037)	0.993 (.030)	0.997 (.028)	1.075 (.036)	1.006 (.058)	1.084 (.067)	0.705 (.052)	0.762 (.060)
mktcapj	1.055 (.035)	1.061 (.032)	1.090 (.031)	1.041 (.033)	1.077 (.058)	1.054 (.062)	0.759 (.052)	0.769 (.054)
sophi	0.456 (.038)	0.610 (.034)	0.466 (.039)	0.411 (.041)	0.627 (.055)	0.423 (.070)	0.542 (.052)	0.442 (.060)
sophj	0.094 (.037)	0.248 (.030)	0.104 (.037)	0.054 (.044)	0.265 (.055)	0.083 (.080)	0.179 (.052)	0.141 (.063)
distij	-	-0.881 (.031)	-0.707 (.039)	-0.663 (.040)	-0.890 (.063)	-0.671 (.078)	-0.824 (.066)	-0.691 (.076)
telephnorij	-	-	0.181 (.027)	0.178 (.027)	-	0.177 (.045)	-	0.133 (.044)
banknorij	-	-	-	0.179 (.045)	-	0.171 (.087)	-	0.058 (.069)
insidersj	-	-	-	-0.018 (.045)	-	0.018 (.085)	-	0.138 (.060)
N	1456	1456	1456	1456	182	182	1456	1456
F (K, N-K-l)	206.71	352.58	334.48	297.16	189.74[b]	132.27[d]	0.077[f]	0.069[f]
R^2	0.555	0.693	0.704	0.707	0.844[c]	0.860[c]	0.823[c]	0.845[c]

a 'Between' regression on group means b F(5,176) c 'Between' d F(8,173) e GLS f R^2 'Within'
In this table as well as all the tables that follow, time dummies are not reported.

Figure 4

	Coeff.	St. Error	t	P>|t|
McapA	1.122	0.128	8.774	0.000
RealConsB	1.068	0.103	10.409	0.000
TranscostA	-0.045	0.015	-3.033	0.003
ReturnA	20.381	2.519	8.091	0.000
Constant	-21.623	1.682	-13.041	0.000

Notes:
The dependent variable is the sales of portfolio equities of country *A* to country *B* (log).
Regression on group means: number of groups = 182
F(4, 77) = 70.41
R-squared "between" = 0.61

Figure 5