SIMON FRASER UNIVERSITY
Department of Economics

Econ 808 — Prof. Kasa
Macroeconomic Theory — Fall 2012

MIDTERM EXAM
(Solutions)

Answer the following questions True, False, or Uncertain. Briefly explain your answers. No credit without explanation. (10 points each).

1. In the standard McCall job search model, higher wage dispersion increases equilibrium unemployment.

   TRUE. *Higher wage dispersion increases the reservation wage. Accepting a job is like exercising an* option. *You don't have to take a job if the offer is lousy. If the tails of the distribution get wider, that means you have a better chance of getting a really good offer, which makes you wait longer. (See either the textbook or the class notes for a formal proof, but a formal proof is not necessary).*

2. If a stochastic process is ergodic, then it must be stationary.

   FALSE. *If a process is ergodic, it has a unique stationary distribution. However, there is no guarantee that the process has reached this distribution. For example, consider the process,* $x_t = \lambda x_{t-1} + \varepsilon_t$*, where* ε_t *is i.i.d.* $N(0, \sigma^2)$*. This process has a unique stationary distribution if* $|\lambda| < 1$*, ie. it's* $N(0, \sigma^2/(1-\lambda^2))$*. However, if the process started at a nonzero value, it might exhibit a drifting mean. This drift could persist for a long time if* λ *is close to one.*

3. The standard Mortensen-Pissarides model generates an equilibrium unemployment rate that is too volatile.

   FALSE. *As discussed in Shimer (AER, 2005), the benchmark MP model generates unemployment and vacancies that are an order of magnitude* less *variable than observed (assuming productivity shocks are calibrated to measured labor productivity, and the value of leisure is calibrated to observed replacement rates).*

   The following questions are short answer. Be sure to explain and interpret your answer. Clarity and conciseness will be rewarded.

4. (15 points). Consider the following expectational difference equation:

$$x_t = \alpha E_t x_{t+1} + \beta z_t \qquad |\alpha| < 1$$

   where the exogenous z_t process evolves according to a stationary AR(1) process: $z_t = \gamma z_{t-1} + \varepsilon_t$, with $|\gamma| < 1$.

   (a) Derive the fundamental solution of this expectational difference equation.

   To calculate the fundamental solution, employ the 'method of undetermined coefficients'. Guess that it takes the form: $x_t = \phi z_t$*. (This is a natural guess, given that* z_t *is AR(1)). Substituting this guess in and matching coefficients we find,* $\phi = \frac{\beta}{1-\alpha\gamma}$*. Hence, the fundamental solution is*

$$x_t = \left(\frac{\beta}{1-\alpha\gamma}\right) z_t$$

(b) Derive the full class of solutions, including any potential bubble solutions.

Bubble solutions solve the homogeneous equation, $x_t = \alpha E_t x_{t+1}$*. One can readily verify that any process satisfying the recursion,* $B_t = \frac{1}{\alpha} B_{t-1} + \epsilon_t$*, where* ϵ_t *is a martingale difference process, satisfies the homogeneous equation. Hence, the full class of solutions is*

$$x_t = \left(\frac{\beta}{1-\alpha\gamma}\right) z_t + \left(\frac{1}{\alpha}\right)^t B_0$$

where B_0 *is an arbitrary constant.*

5. (25 points). Consider an agent whose only source of income is invested wealth. The agent enters period-t with current asset holdings, a_t, and must decide how much to consume, and how much to invest. His preferences are logarithmic,

$$V_0 = \max_{c_t} E_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

with initial assets, a_0, given exogenously. If he consumes c_t this period, then his wealth at the beginning of next period is $a_{t+1} = (1 + r_{t+1})(a_t - c_t)$. The only problem is that the rate of return is unknown at the time the agent chooses this period's consumption. However, the agent does know that $\log(1+r_{t+1})$ is an i.i.d. normally distributed random variable, with mean $(1 + \bar{r})$ and variance σ^2.

(a) Characterize the agent's consumption/savings problem as a dynamic programming problem (ie, write down the Bellman equation). What is the control variable and what is the state variable?

Given the budget constraint, $a' = R'(a - c)$*, we can write the Bellman equation in one of two ways,*

$$V(a) = \max_c \{\log(c) + \beta EV[R'(a - c)]\}$$

where the control variable is consumption, or as

$$V(a) = \max_{a'} \{\log(a - a'/R') + \beta EV(a')\}$$

where the control variable is a'*. In both cases, the relevant state variable is the current level of assets,* a*. (Note:* R *is not a state variable, since* R *is i.i.d., so that the current* R *has not predictive content for next period's value,* R'*).*

(b) Employ a 'guess-and-verify' strategy to solve the agent's Bellman equation. (Hint: Guess that the value function is logarithmic).

Guess that $V(a) = A + B\log(a)$*, where* A *and* B *are undetermined coefficients. The Bellman equation becomes,*

$$A + B\log(a) = \max_c \{\log(c) + \beta E[A + B\log(R'(a - c))]\}$$

The FOC is

$$\frac{1}{c} - \frac{\beta B}{a - c} = 0 \qquad \Rightarrow \qquad c = \frac{a}{1 + \beta B}$$

Substituting this back into the Bellman equation gives us the fixed point condition

$$A + B\log(a) = \log(a) - \log(1 + \beta B) + \beta\{A + B(1 + \bar{r}) + B[\log(\beta B a) - \log(1 + \beta B)]\}$$

Matching coefficients we find, $B = 1/(1 - \beta)$*, from which it follows that the policy function takes the form,* $c = (1 - \beta)a$*. Notice an interesting feature of this policy function - the saving rate is* *completely independent of the expected rate of return! (This owes to a special, knife-edge property of log preferences, which involves exactly offsetting income and substitution effects. We'll have more to say about this when we get to asset pricing). To calculate A we must then substitute our solution for B back in, and then find the constant. Evidently, it's a bit ugly, but not that important, since it has no bearing on observable behavior (although it does influence welfare of course). I suppose I'll have to calculate it when grading, but for now I'm skipping it. (I'm not going to put much weight on getting A right).*

6. (30 points). **Wage Dispersion in the Mortensen-Pissarides Model**. In the MP model discussed in class, workers were identical, and there was no wage dispersion in equilibrium. This question asks you to extend the model in order to generate a simple theory of equilibrium wage dispersion.

   Consider an economy consisting of a unit measure continuum of risk neutral, ex ante identical workers, and a larger measure continuum of ex ante identical firms. Time is discrete and infinite. Workers and firms share a common discount factor β. Workers can search freely, but firms must pay a vacancy cost of c while they are searching for workers. Matching takes place according to a standard constant returns matching function. Let $\mu(\theta_t)$ denote the probability that a worker meets a firm, and $\mu(\theta_t)/\theta_t$ be the probability that a firm meets a worker, where as usual, θ_t denotes labor market 'tightness' in period-t (ie, the measure of vacancies relative to unemployed workers). Assume $\mu(\theta)$ is continuous and twice differentiable, with $\mu''(\theta) < 0 < \mu'(\theta)$. Also assume $\mu(\theta) \leq \min\{\theta, 1\}$.

   In contrast to the model discussed in class, assume now that when a worker and firm are matched, they draw a match-specific productivity, y, from a continuous distribution $F(y)$, with support $[\underline{y}, \overline{y}]$. Assume y is observed by both workers and firms, and is constant during a match. As usual, suppose matches are exogenously destroyed with probability s each period, and that workers and firms set wages according to Nash bargaining. Finally, assume that a worker gets b each period he is not matched.

   (a) Write down the Bellman equations for an unemployed worker (U_t), an employed worker in a match with productivity y, ($V_t(y)$), a firm with an unfilled vacancy (W_t), and an operating firm with productivity y, ($J_t(y)$). Note that all value functions are evaluated at the beginning of the period, before matching and separation occur. Also, let $e_t(y) \in \{0, 1\}$ be an indicator variable denoting whether a job is created after observing y.

   The Bellman equations are

$$
\begin{aligned}
U_t &= \mu(\theta_t)\left(\int [w_t(y) + \beta V_{t+1}(y)]e_t(y) + (1 - e_t(y))(b + \beta U_{t+1})\right) dF(y) + (1 - \mu(\theta_t))(b + \beta U_{t+1}) \\
V_t(y) &= s(b + \beta U_{t+1}) + (1 - s)[w_t(y) + \beta V_{t+1}(y)] \\
W_t &= -c + \frac{\mu(\theta_t)}{\theta_t}\left(\int e_t(y)[y - w_t(y) + \beta J_{t+1}(y)]dF(y)\right) + \left(1 - \frac{\mu(\theta_t)}{\theta_t}\right)\beta \max\{W_{t+1}, 0\} \\
J_t(y) &= s\beta \max\{0, W_{t+1}\} + (1 - s)(y - w_t(y) + \beta J_{t+1}(y))
\end{aligned}
$$

   (b) Now focus on a steady state equilibrium. Write down the Nash bargaining conditions, assuming that all workers have bargaining power η. Show how match surplus is divided. Show that there is a threshold productivity, $\hat{y}$, such that $e(y) = 1$ iff $y \geq \hat{y}$. Derive an explicit expression for $\hat{y}$ in terms of b, β, and U.

   In a steady state we can set time-t value functions equal to their time-$(t+1)$ counterparts, and impose the free entry condition, $W_t = 0$. That significantly simplifies the above Bellman equations. Nash bargaining takes the form

$$
w(y) = \max_{\omega}\left[\omega + \beta\tilde{V}(\omega; y) - b - \beta U\right]^{\eta}\left[y - \omega\beta\tilde{J}(\omega; y)\right]^{1-\eta}
$$

To derive the cutoff productivity, note that given y the steady state match surplus is just

$$S(y) = \frac{y - (1-\beta)(b+\beta U)}{1-(1-s)\beta}$$

where the numerator is the per-period net benefit of the match (ie, value of output less value of free time). Note that the effective discount rate is $(1-s)\beta$, given the possibility that a match is exogenously destroyed. Hence, the threshold productivity is given by, $\hat{y} = (1-\beta)(b+\beta U)$.

(c) Derive an expression for equilibrium wages as a function of productivity, $w(y)$. Using this, characterize a steady state equilibrium by deriving explicit expressions for θ and $\hat{y}$.

Nash bargaining implies that wages are given by

$$w(y) = (1-\beta)(b+\beta U) + \eta[y - (1-\beta)(b+\beta U)]$$

That is, workers earn their reservation value plus a fraction of the total match surplus, where the fraction is determined by their Nash bargaining. All that remains is to express U as a function of the exogenous parameters. The worker's steady state U Bellman equation can be written as

$$(1-\beta)U = b + \mu(\theta)\left(\int_{\hat{y}}^{\bar{y}} [w(y) - (b+\beta U) + \beta V(y)]dF(y)\right)$$

If we sub in the worker's steady state V value function and use the Nash bargaining wage function we can write this as

$$(1-\beta)U = b + \frac{\eta}{1-\eta}\theta c \tag{1}$$

Where we've used the fact that the firm's two value functions imply

$$c = \frac{\mu(\theta)}{\theta}\frac{\int_{\hat{y}}^{\bar{y}}[y - w(y)]dF(y)}{1-\beta(1-s)} \tag{2}$$

which has the usual interpretation of profits paying for search costs. Finally, if we sub eq (1) into the Nash wage equation we get

$$w(y) = (1-\eta)b + \eta(y + \beta\theta c) \tag{3}$$

which is what we want. If we sub this into eq (2) we can then determine the equilibrium value of θ.

(d) Finally, using the above expressions for $\hat{y}$ and $w(y)$, characterize the equilibrium wage distribution as a function of the productivity distribution $F(y)$.

The equilibrium wage distribution is determined by integrating (3) against F, above the threshold value, $\hat{y}$.