SIMON FRASER UNIVERSITY
Department of Economics

Econ 808
Macroeconomic Theory

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PROBLEM SET 1 - Time Series and Dynamic Optimization
(Due October 3)

1. **Solving the Continuous-Time LQR Three Different Ways**

Consider the following dynamic optimization problem:

$$\min_u \int_0^\infty \frac{1}{2}\left(qx^2 + ru^2\right) e^{-\rho t} dt$$

subject to:

$$\begin{aligned} \dot{x} &= ax + bu \qquad\qquad a < 0 \\ x(0) &= \bar{x}_0 \\ \lim_{t\to\infty} x(t)e^{-\rho t} &= 0 \end{aligned}$$

where x is the state variable, u is the control variable, and q and r are parameters reflecting the costs of state and control variability.

(a) Solve this problem using the calculus of variations. That is, derive the Euler equation, which is a second-order ordinary differential equation in x. Use the boundary conditions to determine the unique solution. How do we know we have a minimum rather than a maximum?

(b) Solve this problem using Pontryagin's Maximum Principle. That is, define the (current-value) Hamiltonian, and derive the necessary conditions for an optimum (ie, derive a pair of first-order differential equations in x and the co-state variable for the constraint). Show that the eigenvalues of this system are the same as the roots of the Euler equation derived in part (a). Use the boundary conditions to determine a unique solution.

(c) Solve this problem using dynamic programming. That is, define the Bellman equation, and derive the necessary conditions for an optimum. Posit that the value function is of the form: $V(x) = Px^2$ and use the method of undetermined coefficients to determine a first-order differential equation characterizing P. (This equation is called the 'Riccati equation'). Determine the steady state value of P, and use it to calculate the steady-state optimal feedback policy for u.

(d) Verify that the optimal paths for x and u are the same in parts a, b, and c.

2. **Solving the Discrete-Time LQR Three Different Ways**

Consider the following dynamic optimization problem:

$$\min_{\{u_t\}} \frac{1}{2} \sum_{t=0}^{\infty} \beta^t (q x_t^2 + r u_t^2) \qquad \beta < 1$$

subject to:

$$\begin{aligned} x_{t+1} &= a x_t + b u_t \qquad |a| < 1 \\ x_0 &= \bar{x}_0 \\ \lim_{t \to \infty} \beta^t x_t &= 0 \end{aligned}$$

where x_t is a discrete sequence of state variables, u_t is a discrete sequence of control variables, and q and r are parameters reflecting the costs of state and control variability.

(a) Solve this problem using the calculus of variations. That is, derive the Euler equation, which is now a second-order (linear) difference equation in x_t. Use the boundary conditions to determine the unique solution.

(b) Solve this problem using the maximum principle, i.e., attach a sequence of Lagrange Multipliers to the constraints. Derive a pair of (linear) first-order difference equations in x_t and the Lagrange Multiplier. Show that the eigenvalues of the first-order system are the same as the roots of the characteristic equation in part (a). Use the boundary conditions to determine a unique solution.

(c) Solve this problem using dynamic programming. That is, define the Bellman equation, and use it to characterize an optimal feedback policy for u_t. Posit that the value function is of the form $V(x_t) = \frac{1}{2} P_t x_t^2$, and use the method of undetermined coefficients to calculate a first-order (nonlinear) difference equation for P_t. (This equation is called the Riccati equation). Prove that there is a unique positive steady state solution of the Riccati equation.

(d) Verify that the optimal paths for x_t and u_t are the same in parts a, b, and c.

3. Consider the following "cake-eating" type problem, where an agent must decide how to allocate a fixed amount of total consumption over an infinite horizon:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$\sum_{t=0}^{\infty} c_t \leq s \qquad c_t \geq 0$$

(a) Characterize this problem as a dynamic programming problem. What is the state variable? What is the state transition equation? What is the Bellman equation?

(b) Show that if $\beta = 1$ and $u(c) = \gamma c$ for some $\gamma > 0$, then the problem always has at least one solution. Find a solution.

(c) Show that if $\beta = 1$ and $u(c)$ is strictly increasing and strictly concave then a solution does not exist. If you can't prove it formally, explain intuitively.

(d) Show that if $0 < \beta < 1$ and $u(c) = \sqrt{c}$, then the problem can be solved analytically. In particular, solve it in two different ways:

  (i) Combine the Euler equation with the budget constraint to solve for the optimal sequence of c_t's.

  (ii) Pursue a 'guess and verify' strategy for solving the Bellman equation in part (a). Given the value function, solve for the policy function.

(e) Verify that solutions (i) and (ii) give you the same result.

4. Use the MATLAB program **bigshow.m** to simulate the following time series processes and to report their impulse response functions and their (log) spectral densities.

  (a) $x_t = \varepsilon_t$

  (b) $x_t = \varepsilon_t + .5\varepsilon_{t-1}$

  (c) $x_t = .9x_{t-1} + \varepsilon_t$

  (d) $x_t = 1.2x_{t-1} - .3x_{t-2} + \varepsilon_t$

  (e) $x_t = 1.2x_{t-1} + .3x_{t-2} + \varepsilon_t$

  (f) $x_t = .8x_{t-1} - .5x_{t-2} + \varepsilon_t$

  where ε_t is an iid shock. Briefly comment on the stochastic properties of these processes. Which exhibit business cycle features? Which are stationary? Do any possess interior maxima in their spectral densities? If so, what is the frequency of the implied cycle?

5. Do Exercise 2.14 in Ljungqvist & Sargent.

6. Do Exercises 5.1 in Ljungqvist & Sargent.

7. At the beginning of each period, a worker can choose to work at her last period's wage or draw a new wage. If she draws a new wage, the old wage is lost and she must wait one period before she can start her new job. New wages are i.i.d. draws from the c.d.f. F, where $F(0) = 0$, $F(B) = 1$ for $B < \infty$. The worker seeks to maximize $E_0 \sum_{t=0}^{\infty} \beta^t w_t$, where w_t is the wage in period-t.

  (a) Write down the Bellman equation for the worker. (Hint: the value function appears in both options available to the worker).

  (b) Now suppose the worker receives an unemployment compensation of c. What is the Bellman equation now?

  (c) Assume wage offers are distributed uniformly on the interval $[0, 1]$. Use the program **ex2.m** to numerically solve the Bellman equation for alternative values of β and c. (This program first discretizes things and then uses a value function iteration method. The value function iteration is done by calling the program **valit.m**.) How does the reservation wage vary with β and c? Interpret the results.