Dimensionality Reduction of the Pinning Control Problem for Network Synchronization

Francesco Sorrentino[1], Shirin Panahi[1], Matteo Lodi[2], Marco Storace[2]

[1] The University of New Mexico, Department of Mechanical Engineering.
[2] Università of Genova, Dipartimento di Ingegneria Navale, Elettrica, Elettronica e delle Telecomunicazioni - DITEN.

Outline

- Dynamical Network
  - Definition
  - Representation
- Collective behavior
  - Synchronization
- Control
  - Pinning control

Figure: Starling murmuration [a]

[a]Photo: Donald Macauley

- Dynamical Network
  - Definition: Interactive dynamical units
    - Social
    - Economical
    - Biological
    - etc.
  - Representation: Graph Theory
    - Nodes
    - Links

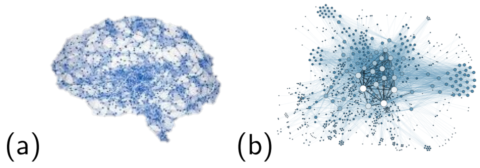


Figure: (a)Neural Network. (b) Graph

Dynamical Network

- Homogeneous,
  - Same dynamical system in each node, AND
  - Same interaction function between the nodes
- Heterogeneous
  1. Different Dynamical System in each nodes, OR
  2. Different interaction function between the nodes

Synchronization

- A general equation for a homogeneous network of coupled dynamical systems

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{F}(\boldsymbol{x}_i(t)) + \sigma \sum_{j=1}^{N} a_{ij} \boldsymbol{H}(\boldsymbol{x}_j(t)) \quad i = 1, \cdots, N \tag{1}$$

  - $\boldsymbol{x}_i$: the m-dimensional state vector of the i^{th} dynamical system (oscillator),
  - $\boldsymbol{F}$: the dynamics of each oscillator,
  - A: the connectivity matrix (topology of the network connections)
  - σ: the overall coupling strength
  - $\boldsymbol{H}$: the output function

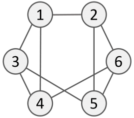
Synchronization

$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{F}(\boldsymbol{x}_i(t)) + \sigma \sum_{j=1}^{N} a_{ij} \boldsymbol{H}(\boldsymbol{x}_j(t)) \quad i = 1, \cdots, N$

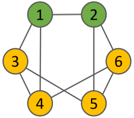
(a) Complete synchronization in homogeneous networks
- Matrix A: Constant row sum (Usually Laplacian matrix L)
- Synchronization manifold: $\boldsymbol{s}(t) = \boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_N$

(b) Cluster synchronization
- Cluster Synchronization of Networks via a Canonical Transformation for Simultaneous Block Diagonalization of Matrices

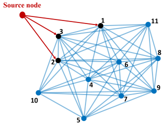
(c) Complete synchronization in heterogeneous networks (links of different types)
- Pinning control of networks: dimensionality reduction through simultaneous block-diagonalization of matrices



(a)



(b)



(c)

Complete Synchronization

Homogeneous network

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{F}(\boldsymbol{x}_i(t)) + \sigma \sum_{j=1}^{N} l_{ij} \boldsymbol{H}(\boldsymbol{x}_j(t)) \quad i = 1, \cdots, N \tag{2}$$

- Synchronization,
  - Complete Synchronization,
  - Matrix L: Laplacian matrix (zero-row sum)
  - Synchronization manifold: $\boldsymbol{s}(t) = \boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_N$
- Stability analysis
  - Dynamics on synchronous manifold: $\dot{\boldsymbol{s}}(t) = \boldsymbol{F}(\boldsymbol{s}(t))$.
- Is this manifold stable?

Stability analysis

Master Stability Approach

- $(\boldsymbol{x}_i(t) = \boldsymbol{s}(t) + \delta\boldsymbol{x}_i(t))$
- $\lim_{t\to\infty} \delta\boldsymbol{x}_i(t) = 0$
- Linearized system of equations

$$\delta\dot{\boldsymbol{x}}_i(t) = D\boldsymbol{F}(\boldsymbol{s}(t))\delta\boldsymbol{x}_i(t) + \sigma\sum_{j=1}^{N} l_{ij}\boldsymbol{H}(\boldsymbol{s}(t))\delta\boldsymbol{x}_j(t) \quad i = 1, \cdots, N \tag{3}$$

- Master Stability approach (Pecora & Caroll (1998))
  - Diagonalizing the Laplacian matrix L using eigenvalue decomposition
  - $\boldsymbol{\eta}(t) = (V \otimes I_m)\delta\boldsymbol{x}(t)$

- Master stability equations: a set of independently evolving perturbations:

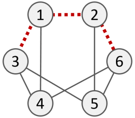
$$\dot{\boldsymbol{\eta}}_j(t) = [D\boldsymbol{F}(\boldsymbol{s}(t)) - \sigma\lambda_j D\boldsymbol{H}(\boldsymbol{s}(t))]\boldsymbol{\eta}_j(t), \tag{4}$$

  where

  - $\begin{cases} \lambda_1 = 0 & \textit{parallel perturbation} \\ \lambda_j \quad j = 2, \cdots, N & \textit{transverse perturbation} \end{cases}$

- MSF $\mathcal{M}(\sigma\lambda_i)$: The largest transverse Lyapunov exponent as a function of the parameter $\sigma\lambda_i$.

- The case of a heterogeneous networks with different types of interactions was first considered by F. Sorrentino in this paper: "*Synchronization of hypernetworks of coupled dynamical systems, (2012)*"



$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t)) + \sigma_A \sum_{j=1}^{N} l_{ij}^{A} \mathbf{H}_A(\mathbf{x}_j(t)) + \sigma_B \sum_{j=1}^{N} l_{ij}^{B} \mathbf{H}_B(\mathbf{x}_j(t))$$

$$\delta\dot{\mathbf{x}}_i(t) = D\mathbf{F}(\mathbf{s}(t))\delta\mathbf{x}_i + \sigma_A \sum_{j=1}^{N} l_{ij}^{A} D\mathbf{H}_A(\mathbf{s}(t))\delta\mathbf{x}_j + \sigma_B \sum_{j=1}^{N} l_{ij}^{B} D\mathbf{H}_B(\mathbf{s}(t))\delta\mathbf{x}_j \tag{5}$$

- The two Laplacian matrices L^A and L^B commute
- One of the networks (either L^A or L^B) is unweighted and fully connected
- One of the two networks (say, e.g., L^A) is such that $L_{ij}^{A} = a_j$ for $i, j = 1, \cdots, N$

Simultaneous Block Diagonalization

Given a set of $n \times n$ real matrices $\{A_1, A_2, \cdots, A_N\}$, find an $n \times n$ orthogonal matrix P such that $P^T A_1 P, \cdots, P^T A_N P$ are in a common block-diagonal form.

- Maehara, T., Murota, K. (2011)

$\mathcal{M}_n$:= Set of $n \times n$ matrices $(\{A_1, A_2, \cdots, A_N\})$

Definition: A matrix $*$-algebra (subalgebra) T:

Subset of $\mathcal{M}_n$ such that if

1) $I_n \in T$ AND
2) $A, B \in T$ AND
3) $\alpha, \beta \in R$

   then $\Rightarrow \alpha A + \beta B,\ AB,\ A^T \in T$

Definition: T': Commutant algebra of T:

1) forms a matrix $*$-algebra
2) contains set of all matrices X that commute with every member of T

Simultaneous Block Diagonalization

- **Theorem** There exist an orthogonal matrix Q such that:

$$Q^T T Q = \oplus_l^N T_j, \tag{6}$$

where each T_j is not further reducible.

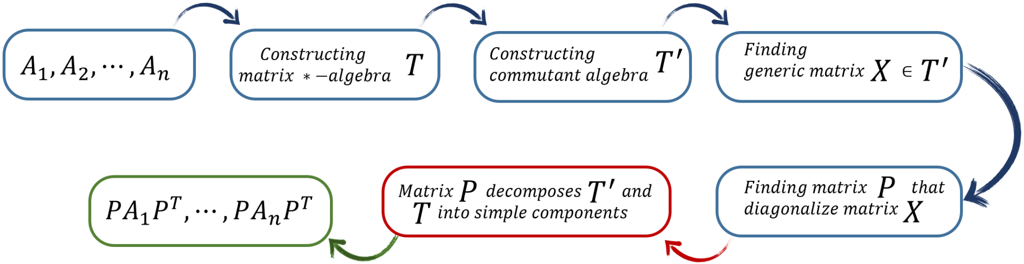
- **Definition::**Generic matrix If we sample a symmetric matrix $A \in T$ randomly, then A is generic with probability one.
- **Proposition.** Let $A \in T$ be a generic symmetric matrix. Then an orthogonal matrix that diagonalizes A decomposes T into simple components.

SBD

- **Proposition.** If an orthogonal matrix Q decomposes a matrix $*$-algebra T into the simple components then Q also decomposes the commutant algebra T' into the simple components.

* To find Q we need to sample a generic matrix X in T'

SBD Overview



Finding matrix P

- A matrix X needs to simultaneously commute with each one of the matrices A_k which means $[A_k, X] = A_k X - X A_k = O_n$, $k = 1, \dots, M$,
- Define the vectorizing function $\text{vec} : \mathbb{R}^{n \times m} \mapsto \mathbb{R}^{nm}$, we have:

$$\text{vec}(A_k X - X A_k) = \text{vec}(A_k X) - \text{vec}(X A_k) = \text{vec}(O_n), \quad k = 1, 2, \dots, M \quad (7)$$

- To find the matrix X:
  - Define matrix $Y_k = (I_n \otimes A_k - (A_k)^T \otimes I_n)$
  - Look for a vector $\text{vec}(X) \in \bigcap_{k=1}^{M} \mathcal{N}(Y_k)$
- By calculating the matrix X, the matrix P is constructed as the eigenvectors of the matrix X.

Application of SBD to Synchronization of Networks

- Irving, D., Sorrentino, F. (**2012**)
  - Complete Synchronization of Heterogeneous Networks
- Zhang, Y., Motter, A. E. (**2020**)
  - Cluster Synchronization
- Panahi, S., Klickstein, I., Sorrentino, F. (**2021**).
  - Canonical Transformation
- Panahi, S., Lodi, M., Storace, M., Klickstein, I., Novello, G., Sorrentino, F.
  - Pinning control problem

SBD & Cluster Synchronization

"Cluster Synchronization of Networks via a Canonical Transformation for Simultaneous Block Diagonalization of Matrices, Chaos, 2021"

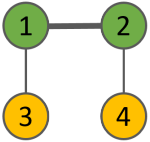
- **Goal**: Proposing a canonical transformation for simultaneous block diagonalization of matrices, applicable for investigating the cluster synchronization of networks
- **Advantages**:
  1) Decouple the stability problem into subproblems of minimal dimensionality while preserving physically meaningful information
  2) Applicable for both orbital and equitable partitions of the network nodes
  3) Parametrization of the problem in a small number of parameters

Cluster synchronization

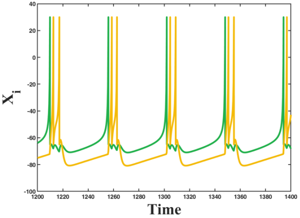
Given a network with adjacency matrix A:

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{F}(\boldsymbol{x}_i(t)) + \sum_{j=1}^{N} a_{ij}\boldsymbol{H}(\boldsymbol{x}_j(t)) \quad i = 1, \cdots, N$$

- **Clusters**: The network nodes ($\mathcal{V}$) can be partitioned into C numbers of clusters $\mathcal{C}_1, \mathcal{C}_2, .., \mathcal{C}_C$, $\cup_{k=1}^{C} \mathcal{C}_k = \mathcal{V}$, $\mathcal{C}_k \cap \mathcal{C}_\ell = \emptyset$ for $k \neq \ell$,
- **Coloring of the Nodes**: Partitioning of the nodes of the network induces a colored network, where each node i is assigned a color k if node i is in cluster $\mathcal{C}_k$.
- **Cluster Synchronization**: Nodes in each cluster synchronize on the same time evolution but these time evolutions are different for nodes in different clusters

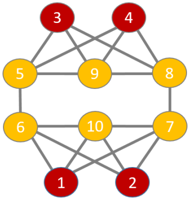


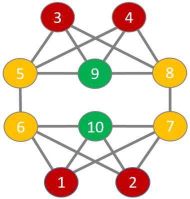
$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



Cluster synchronization

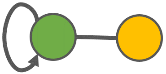
- **Equitable Clusters**: all the nodes in one cluster has exactly the same number of neighbors in the other clusters regardless of the choice of the nodes
- **Orbital Clusters**: Nodes in the same orbital cluster can be swapped with each other without changing the overall network topology.





Cluster synchronization

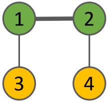
- **Quotient network**: Describes a network for which all the nodes in each cluster collapse to a single quotient node.



$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- **Quotient matrix**: $Q_{kl} = \sum_{j \in \mathcal{C}_l} A_{ij} \quad i \in \mathcal{C}_k$.
- **Cluster synchronization dynamics**:
$\dot{\boldsymbol{s}}_k(t) = \boldsymbol{F}(\boldsymbol{s}_k(t)) + \sum_{l=1}^{C} Q_{kl} \boldsymbol{H}(\boldsymbol{s}_l(t)), \quad k, l = 1, 2, \cdots, C,$

- **Cluster Indicator matrix**: $E_k = \{e_{k_{ij}}\}_{N \times N}$, $e_{k_{ii}} = 1$ if node $i \in \mathcal{C}_k$, otherwise 0.



$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Stability Analysis

- $\delta \boldsymbol{x}_i = (\boldsymbol{x}_i - \boldsymbol{s}_k)$, $i \in \mathcal{C}_k$

$$\delta \dot{\boldsymbol{x}}(t) = \left[\sum_{c=1}^{C} E_c \otimes D\boldsymbol{F}(\boldsymbol{s}_c(t)) + \sum_{c=1}^{C} AE_c \otimes D\boldsymbol{H}(\boldsymbol{s}_c(t)) \right] \delta \boldsymbol{x}(t), \qquad (8)$$

- Is it possible to reduce the mN-dimensional stability problem into a set of independent lower-dimensional equations?
- It is needed to simultaneously block diagonalize the set of $C+1$ matrices $\{A, E_1, E_2, \cdots, E_C\}$
    - IRR transformation (Pecora, L. M., Sorrentino, F., Hagerstrom, A. M., Murphy, T. E., & Roy, R. (2014)) $\Rightarrow$ Orbital clusters
    - SBD technique (Zhang, Y., & Motter, A. E. (2020))

Canonical SBD

Finding matrix P

- **Assumption:** Without loss of generality $\Rightarrow$ we can order the network nodes so that:

$$E_1 = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0_{N-n_1} \end{pmatrix} \quad E_2 = \begin{pmatrix} 0_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0_{N-(n_1+n_2)} \end{pmatrix} \quad \cdots \quad E_C = \begin{pmatrix} 0_{N-n_C} & 0 \\ 0 & I_{n_C} \end{pmatrix} \tag{9}$$

- Lemma 1. Any matrix P that commutes with the set of matrices E_i has the following block-diagonal structure,

$$P = \begin{pmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_C \end{pmatrix} \tag{10}$$

Canonical SBD

- The transformation matrix T:
  - the eigenvectors of the matrix P for its columns
  - must also have the same block-diagonal structure as the matrix P

$$T = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_C \end{pmatrix} \quad (11)$$

- $T^T = T^{-1} \Rightarrow T_k^T = T_k^{-1}$
- Each block T_k corresponds to each cluster $\mathcal{C}_k \Rightarrow$ each block describes stability of either one cluster or a set of intertwined clusters
- After applying T, $\exists$ one block corresponds to the quotient dynamics.
- $J_i = TE_iT^T = E_i$ for $i = 1, 2, \cdots, C \quad \Rightarrow$ T is 'Canonical'

Canonical SBD

- After applying T, we obtain,

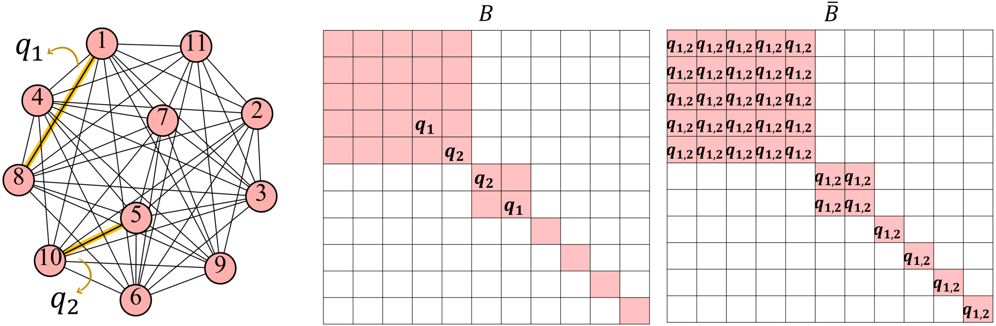
$$\dot{\boldsymbol{\eta}}(t) = \left[\sum_{k=1}^{C} J_k \otimes D\boldsymbol{F}(\boldsymbol{s}_k(t)) + \sum_{k=1}^{C} BJ_k \otimes D\boldsymbol{H}(\boldsymbol{s}_k(t))\right] \boldsymbol{\eta}(t), \qquad (12)$$

$$\dot{\mathbf{Y}} = T^{-1}\delta\dot{\mathbf{X}} = \frac{\sqrt{2}}{2}\begin{pmatrix} \delta\dot{\boldsymbol{x}}_1 + \delta\dot{\boldsymbol{x}}_2 \\ -\delta\dot{\boldsymbol{x}}_3 - \delta\dot{\boldsymbol{x}}_4 \\ \delta\dot{\boldsymbol{x}}_1 - \delta\dot{\boldsymbol{x}}_2 \\ \delta\dot{\boldsymbol{x}}_3 - \delta\dot{\boldsymbol{x}}_4 \end{pmatrix} = \begin{pmatrix} D\boldsymbol{F}(\boldsymbol{s}_1) & 0 & 0 & 0 \\ 0 & D\boldsymbol{F}(\boldsymbol{s}_2) & 0 & 0 \\ 0 & 0 & D\boldsymbol{F}(\boldsymbol{s}_1) & 0 \\ 0 & 0 & 0 & D\boldsymbol{F}(\boldsymbol{s}_2) \end{pmatrix} \mathbf{Y}$$

$$+ \begin{pmatrix} 0 & -D\boldsymbol{H}(\boldsymbol{s}_2) & 0 & 0 \\ -D\boldsymbol{H}(\boldsymbol{s}_1) & D\boldsymbol{H}(\boldsymbol{s}_2) & 0 & 0 \\ 0 & 0 & 0 & D\boldsymbol{H}(\boldsymbol{s}_2) \\ 0 & 0 & D\boldsymbol{H}(\boldsymbol{s}_1) & -D\boldsymbol{H}(\boldsymbol{s}_2) \end{pmatrix} \mathbf{Y},$$

Advantages:

1) Compared to previous work [Zhang and Motter (2020)] it has lower number of nonzero entries that parametrize the matrices after the transformation



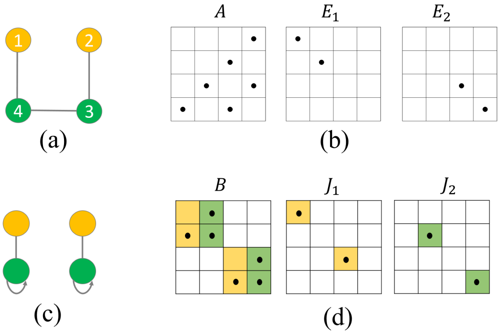
(a) (b) (c)

Canonical SBD

Advantages:

2) Each network node after the transformation belongs to one and only one cluster

- Canonical SBD



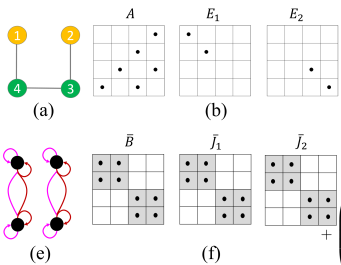
$$\dot{\mathbf{Y}} = T^{-1}\delta\dot{\mathbf{X}} = \frac{\sqrt{2}}{2}\begin{pmatrix} \delta\dot{\mathbf{x}}_1 + \delta\dot{\mathbf{x}}_2 \\ -\delta\dot{\mathbf{x}}_3 - \delta\dot{\mathbf{x}}_4 \\ \delta\dot{\mathbf{x}}_1 - \delta\dot{\mathbf{x}}_2 \\ \delta\dot{\mathbf{x}}_3 - \delta\dot{\mathbf{x}}_4 \end{pmatrix} = \begin{pmatrix} DF(\mathbf{s}_1) & 0 & 0 & 0 \\ 0 & DF(\mathbf{s}_2) & 0 & 0 \\ 0 & 0 & DF(\mathbf{s}_1) & 0 \\ 0 & 0 & 0 & DF(\mathbf{s}_2) \end{pmatrix}\mathbf{Y}$$

$$+ \begin{pmatrix} 0 & -DH(\mathbf{s}_2) & 0 & 0 \\ -DH(\mathbf{s}_1) & DH(\mathbf{s}_2) & 0 & 0 \\ 0 & 0 & 0 & DH(\mathbf{s}_2) \\ 0 & 0 & DH(\mathbf{s}_1) & -DH(\mathbf{s}_2) \end{pmatrix}\mathbf{Y},$$

Canonical SBD

$$\dot{\boldsymbol{\eta}}(t) = \left[\sum_{k=1}^{C} J_k \otimes \boldsymbol{DF}(\boldsymbol{s}_k(t)) + \sum_{k=1}^{C} BJ_k \otimes \boldsymbol{DH}(\boldsymbol{s}_k(t))\right] \boldsymbol{\eta}(t),$$

- $\tilde{T}$ SBD transformation



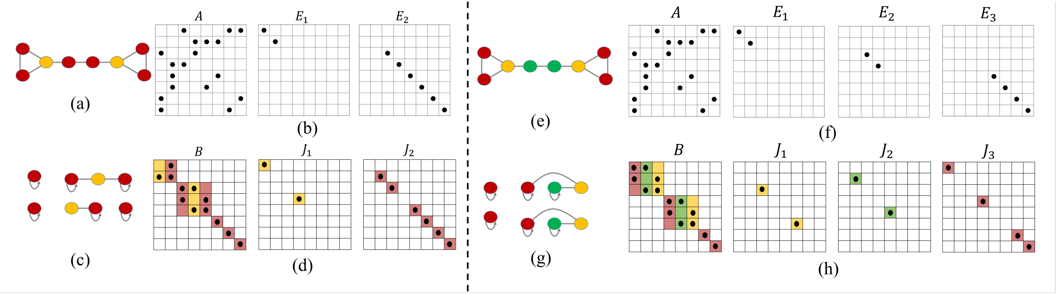
$$\dot{\mathbf{Z}} = \tilde{T}^{-1}\delta\dot{\mathbf{X}} = \begin{pmatrix} -0.65\delta\dot{\boldsymbol{x}}_1 + 0.65\delta\dot{\boldsymbol{x}}_2 - 0.26\delta\dot{\boldsymbol{x}}_3 + 0.26\delta\dot{\boldsymbol{x}}_4 \\ -0.26\delta\dot{\boldsymbol{x}}_1 + 0.26\delta\dot{\boldsymbol{x}}_2 + 0.65\delta\dot{\boldsymbol{x}}_3 - 0.65\delta\dot{\boldsymbol{x}}_4 \\ 0.69\delta\dot{\boldsymbol{x}}_1 + 0.69\delta\dot{\boldsymbol{x}}_2 - 0.16\delta\dot{\boldsymbol{x}}_3 - 0.16\delta\dot{\boldsymbol{x}}_4 \\ 0.16\delta\dot{\boldsymbol{x}}_1 + 0.16\delta\dot{\boldsymbol{x}}_2 + 0.69\delta\dot{\boldsymbol{x}}_3 + 0.69\delta\dot{\boldsymbol{x}}_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.14DF(s_1) + 0.86DF(s_2) & 0.35DF(s_1) - 0.35DF(s_2) & 0 & 0 \\ 0.35DF(s_1) - 0.35DF(s_2) & 0.86DF(s_1) + 0.14DF(s_2) & 0 & 0 \\ 0 & 0 & 0.05DF(s_1) + 0.95DF(s_2) & 0.22DF(s_1) - 0.22DF(s_2) \\ 0 & 0 & 0.22DF(s_1) - 0.22DF(s_2) & 0.95DF(s_1) + 0.05DF(s_2) \end{pmatrix} \mathbf{Z}$$

$$+ \begin{pmatrix} 0.35DH(s_1) - 0.51DH(s_2) & 0.86DH(s_1) + 0.20DH(s_2) & 0 & 0 \\ -0.14DH(s_1) + 1.2DH(s_2) & -0.35DH(s_1) - 0.49DH(s_2) & 0 & 0 \\ 0 & 0 & 0.22DH(s_1) + 1.17DH(s_2) & 0.95DH(s_1) - 0.27DH(s_2) \\ 0 & 0 & -0.05DH(s_1) + 0.72DH(s_2) & -0.22DH(s_1) - 0.17DH(s_2) \end{pmatrix} \mathbf{Z},$$

Advantages:

3) It is applicable for both orbital and equitable partitioning of the network nodes.



Application of the Canonical Transformation in Real Networks

Table: Real networks analysis. N is the number of nodes, E the number of edges, N_{ntc} is the number of nontrivial clusters, $\max(|n_c|)$ is the size of the largest equitable cluster. We include the average runtime in seconds for calculation of the transformation matrix T and of the transformation matrix $\tilde{T}$ using the code from [Zhang and Motter (2020)].

Name	N	E	N_{ntc}	$\max(\lvert n_c \rvert)$	Average Runtime for $\tilde{T}$	Average Runtime for T
ca-netscience: Scientist Collaboration Network	379	914	70	6	25.0745	6.4077
Chilean Power Grid Network	218	527	29	7	2.9198	1.9107
Power Grid Network of Western Germany	491	665	43	5	41.7946	14.4780
celegans-dir: Metabolic Network	453	2025	28	4	24.6675	12.6427
Us Airline	332	2126	31	12	11.0195	5.8412
Erdos971	429	1312	20	3	16.92	10.0446
bio-diseasome: Biological Network	516	1188	95	6	90.5763	13.9930
fb-forum: Social network	899	7036	16	5	171.0608	65.6710

SBD and Pinning Control

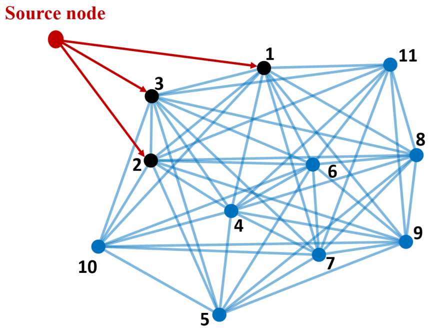
Pinning control of networks: dimensionality reduction through simultaneous block-diagonalization of matrices

Considering pinning control signals : $\mathbf{u}_i(t)$

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{F}(\boldsymbol{x}_i(t)) + \sum_{j=1}^{N} L_{ij}\boldsymbol{G}(\boldsymbol{x}_j(t)) + \mathbf{u}_i(t), \quad (13a)$$

$$\mathbf{u}_i(t) = \gamma r_i[\boldsymbol{H}(\boldsymbol{x}_t(t)) - \boldsymbol{H}(\boldsymbol{x}_i(t))], \quad (13b)$$

$$i = 1, \cdots, N$$



- Previous work (see e.g., Sorrentino, F., Di Bernardo, M., Garofalo, F., Chen, G. (2007)) has dealt with $\boldsymbol{H} = \boldsymbol{G}$.

Synchronization

- Completely synchronized solution $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = \cdots = \boldsymbol{x}_n(t) = \boldsymbol{x}_s(t)$, which obeys,
$$\dot{\boldsymbol{x}}_s(t) = \boldsymbol{F}(\boldsymbol{x}_s(t)). \tag{14}$$
- 'Target' synchronous solution, $\mathbf{x}_t(t)$ produced by the initial condition $\boldsymbol{x}_t^0$,
$$\dot{\boldsymbol{x}}_t(t) = \boldsymbol{F}(\boldsymbol{x}_t(t)), \qquad \boldsymbol{x}_t(0) = \boldsymbol{x}_t^0. \tag{15}$$
- Considering small perturbations $\delta \boldsymbol{x}_i = (\boldsymbol{x}_i - \boldsymbol{x}_t)$ and stacking them together in one vector:
$$\begin{aligned} \delta \dot{\boldsymbol{X}}(t) = [&I_N \otimes D\boldsymbol{F}(\boldsymbol{x}_t(t)) + L \otimes D\boldsymbol{G}(\boldsymbol{x}_t(t)) \\ &-\gamma R \otimes D\boldsymbol{H}(\boldsymbol{x}_t(t))]\delta \boldsymbol{X}(t), \end{aligned} \tag{16}$$

SBD and Pining control problem

$\delta\dot{\boldsymbol{X}}(t) = [I_N \otimes D\boldsymbol{F}(\boldsymbol{x}_t(t)) + L \otimes D\boldsymbol{G}(\boldsymbol{x}_t(t)) - \gamma R \otimes D\boldsymbol{H}(\boldsymbol{x}_t(t))]\delta\boldsymbol{X}(t).$

We need to find transformation matrix $T = SBD(R, L)$

- Finding matrix P:
    - matrix R can be rewritten,

$$R = \begin{pmatrix} I_s & 0 \\ 0 & \mathbf{0}_{N-s} \end{pmatrix}, \tag{17}$$

    - The matrix P has has the following block-diagonal structure,

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \tag{18}$$

- Transformation matrix T

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \tag{19}$$

Driven and Undriven Blocks

- By applying Transformation T to the pair (L, R), we have:

$$T^{-1}LT = L_T = \oplus_{j=1}^{l}\hat{L}_j, \\ T^{-1}RT = R_T = \oplus_{j=1}^{l}\hat{R}_j, \tag{20}$$

- L_T and R_T are block-diagonal:

$$L_T = \begin{pmatrix} L_d & 0 \\ 0 & L_{ud} \end{pmatrix} \qquad R_T = \begin{pmatrix} R_d & 0 \\ 0 & 0_{(N-c)} \end{pmatrix}, \tag{21}$$

- Equation (16) can be split into the one driven and one undriven equations,

$$\dot{\boldsymbol{\eta}}_d(t) = [I_c \otimes D\boldsymbol{F}(\boldsymbol{x}_s(t)) + L_d \otimes D\boldsymbol{G}(\boldsymbol{x}_s(t)) - \gamma R_d \otimes D\boldsymbol{H}(\boldsymbol{x}_s(t))]\boldsymbol{\eta}_d(t),$$

$$\dot{\boldsymbol{\eta}}_{ud}(t) = [I_{N-c} \otimes D\boldsymbol{F}(\boldsymbol{x}_s(t)) + L_{ud} \otimes D\boldsymbol{G}(\boldsymbol{x}_s(t))]\,\boldsymbol{\eta}_{ud}(t). \tag{22}$$

- Dimension of the driven pair (L_d, R_d) = the rank of the controllability matrix of the pair (L, R).

Transformation $\hat{T}$

- $\hat{T} = T_c T_q$
  - T_c : Decouples the problem (16) into a controllable equation and an uncontrollable equation
  - T_q : Decouples the problem (16) into a quotient equation and a redundant equation.
- By applying $\hat{T}$ to the pair (L, R), we have:

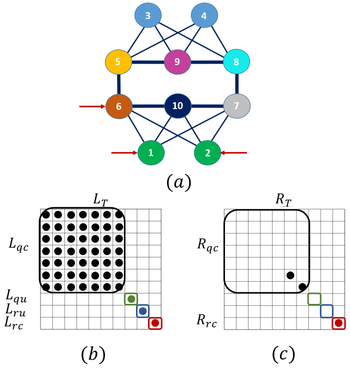
$$\hat{T} L \hat{T}^{-1} = \begin{pmatrix} L_{qc} & 0 & 0 & 0 \\ 0 & L_{rc} & 0 & 0 \\ 0 & 0 & L_{qu} & 0 \\ 0 & 0 & 0 & L_{ru} \end{pmatrix} \hat{T} R \hat{T}^{-1} = \begin{pmatrix} R_{qc} & 0 & 0 & 0 \\ 0 & R_{rc} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (23)$$

Application of transformation $\hat{T}$

- The pair (L,R) is decoupled by application of the transformation $\hat{T}$ into four blocks,

Pair	Type	Size
(L_{qc}, R_{qc})	Quotient controllable block (qc)	u
(L_{rc}, R_{rc})	Redundant controllable block (rc)	$c-u$
$(L_{qu}, 0)$	Quotient uncontrollable block (qu)	$N-c$
$(L_{ru}, 0)$	Redundant uncontrollable block (ru)	

- The transformation $\hat{T}$ leads to a finest SBD of the pair (L,R).



Pinned node selection and multiple driven blocks

- Dimension of the driven pair is affected by
  - Topology of the network
  - Number and choice of the pinned nodes
- it is possible to have the multiple driven pairs (L_d^1, R_d^1), (L_d^2, R_d^2),..., (L_d^w, R_d^w), where $L_d \oplus_{i=1}^{w} L_d^i$ and $R_d \oplus_{i=1}^{w} R_d^i$.
  - A pinned node symmetry (PNS) is a symmetry of the pair (L, R) i.e. a permutation matrix Π that
    - commutes with both matrices L and R
    - it swaps two or more pinned nodes
  - Choosing two or more pinned nodes which belong to the same PNS results into independent driven pairs for each one of these pinned nodes.

Conclusion

- Application of SBD to Synchronization of Dynamical Networks
  - Canonical SBD Transformation on Cluster Synchronization
    - MATLAB code to compute the canonical SBD for the cluster synchronization is available online and can be accessed from https://github.com/SPanahi/Clustered-SBD
  - SBD & Pinning Control Problem in Heterogeneous Networks

Thank you!