Traffic Analysis and System Identification with Legendre Transform

Department of Electrical Engineering, Kyoto University

Takashi HISAKADO

1. Introduction — Objective and Tools

Objective

- Traffic analysis in communication network
- Identification of communication network system

Network system is nonlinear.

$\Longrightarrow$ The analysis is very complicated.

Tools

- Max-plus algebra
- Legendre transform

1. Introduction — Linear description of nonlinear system

Max-plus algebra is defined by

addition $:\oplus \equiv \max$ and multiplication $:\otimes \equiv +$.

We can obtain linear description of nonlinear systems;

- **Discrete event systems**
- **Dynamic programming**
- **Markov decision processes (Viterbi algorithm)**
- **Morphological operations**

etc

TCP network is also max-plus linear.

1. Introduction — Our project

For the system described with max-plus algebra, Legendre transform is a powerful tool.

Legendre transform in max-plus algebra
$\Longrightarrow$ Fourier transform and z-transform in linear system

Legendre transform gives

- **Time series analysis**
  $\Rightarrow$ Traffic analysis
- **Deconvolution**
  $\Rightarrow$ System identification

Our objective is to clarify complicated traffic and to identify complicated network system with Legendre transform.

2. Max-plus algebra — Definition

- **Addition** $\oplus$

$$\oplus \equiv \max$$

- **Multiplication** $\otimes$

$$\otimes \equiv +$$

Example

$$\begin{aligned} 1 \oplus 2 &= \max(1,2) = 2 \\ 2 \otimes 3 &= 2 + 3 = 5 \end{aligned}$$

Difference equations

$$\boldsymbol{x}(t+1) = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

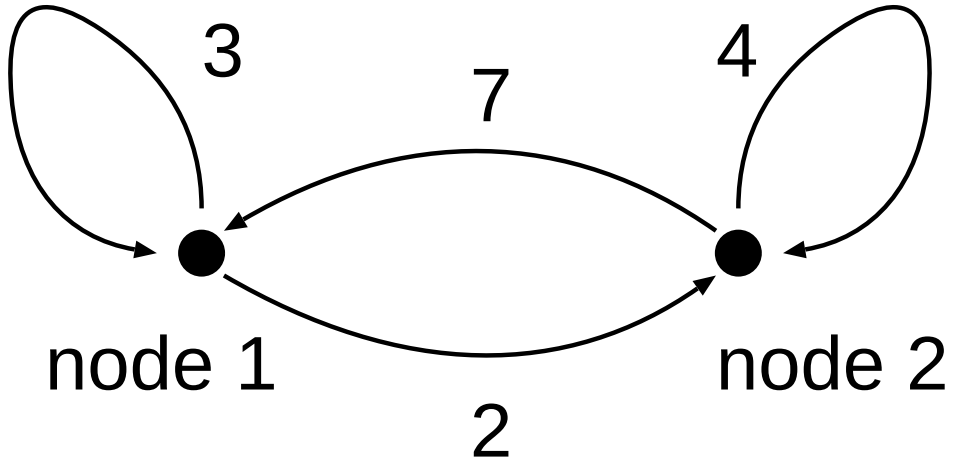
Conventional difference equations

$$\boldsymbol{x}(1) = \begin{pmatrix} 3 \times 1 + 7 \times 0 \\ 2 \times 1 + 4 \times 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}(2) = \begin{pmatrix} 23 \\ 14 \end{pmatrix}, \cdots$$

Max-plus difference equations

$$\boldsymbol{x}(1) = \begin{pmatrix} \max(3+1, 7+0) \\ \max(2+1, 4+0) \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}, \quad \boldsymbol{x}(2) = \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \cdots$$

2. Max-plus algebra — Simple example



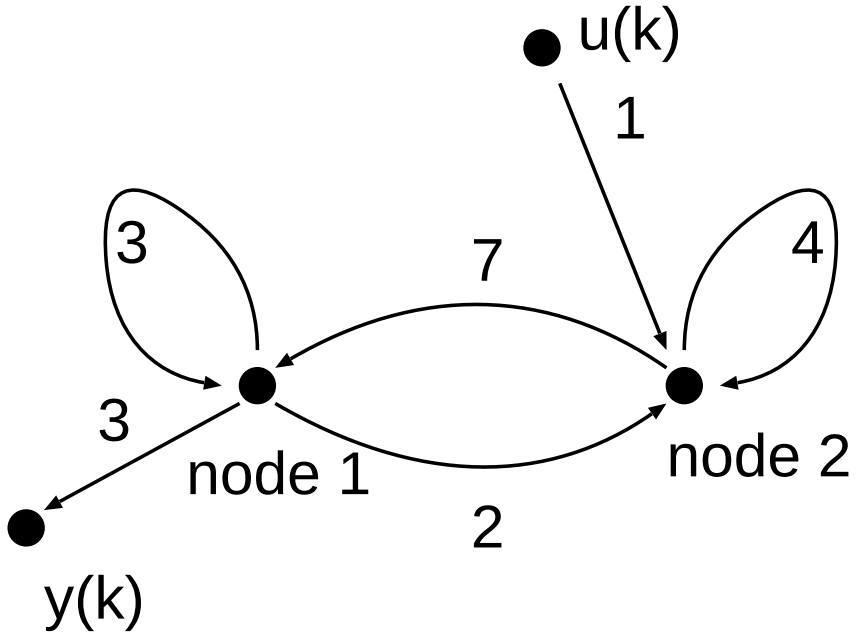
$x_i(k)$ **is the earliest epoch when node** i **becomes active for the** k**-th time.**

A_{ij} **is the traveling time from node** j **to node** i**.**

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} \max(3 + x_1(k), 7 + x_2(k)) \\ \max(2 + x_1(k), 4 + x_2(k)) \end{pmatrix} = \boldsymbol{A}\boldsymbol{x}(k)$$

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}(1) = \begin{pmatrix} 7 \\ 4 \end{pmatrix}, \quad \boldsymbol{x}(2) = \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \cdots$$

2. Max-plus algebra — Example with input and output



$$x(k+1) = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} x(k) \oplus \begin{pmatrix} \epsilon \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 3 & \epsilon \end{pmatrix} x(k)$$

$\epsilon \equiv -\infty$

2. Max-plus algebra — Definition of neutral element

Neutral element with respect to maximization

$$\epsilon \equiv -\infty, \qquad a \oplus \epsilon = a$$

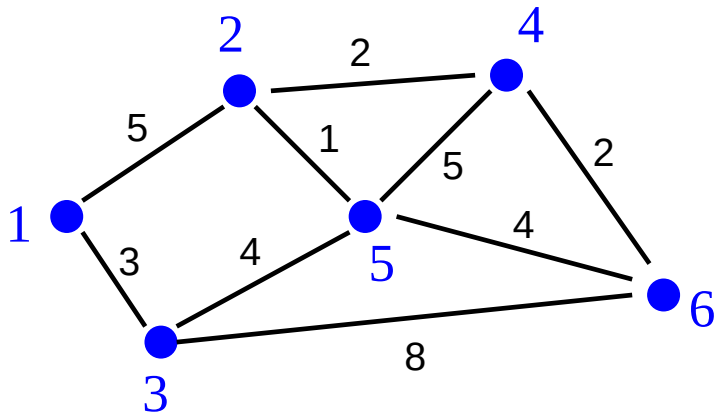
Neutral element with respect to addition

$$e \equiv 0, \qquad a \otimes e = a$$

Relation between conventional and max-plus algebra

$$\begin{aligned} -\infty &= \log 0 \\ 0 &= \log 1 \end{aligned}$$

3. Modeling — Shortest path problem



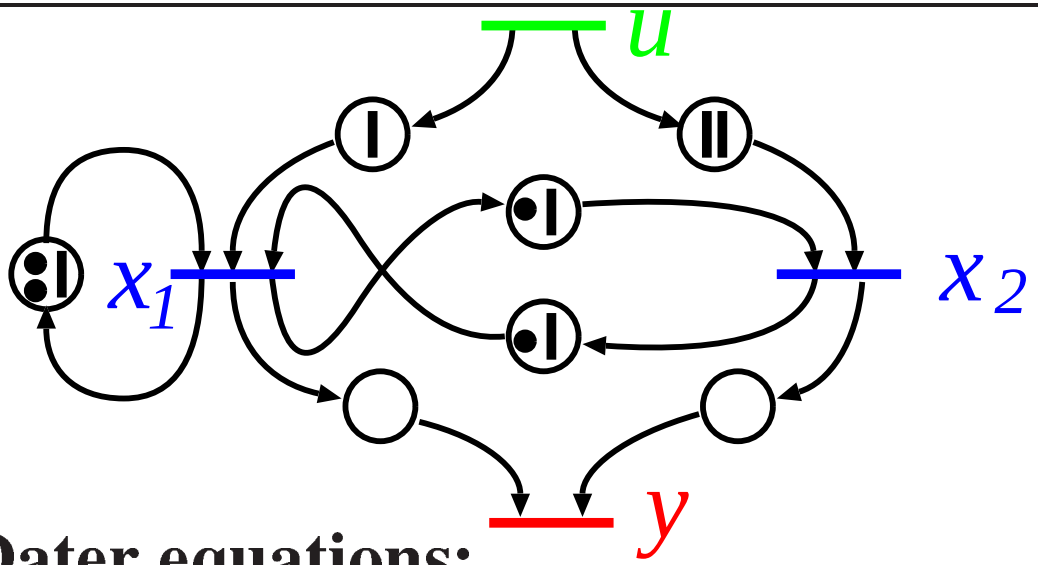
$X_{ij}(k)$ represents the shortest path from j to i with k-path

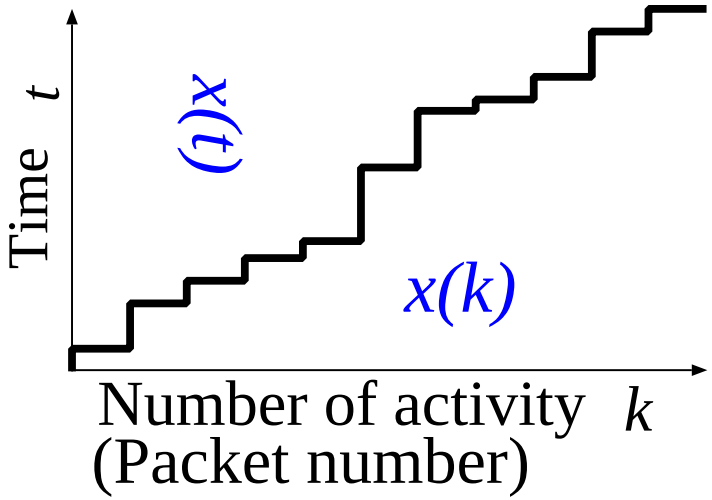
$$X_{ij}(k+1) = \min_{l=1,\ldots,6}(A_{il} + X_{lj}(k)), \quad A = \begin{pmatrix} \epsilon & 5 & 3 & \epsilon & \epsilon & \epsilon \\ 5 & \epsilon & \epsilon & 2 & 1 & \epsilon \\ 3 & \epsilon & \epsilon & \epsilon & 4 & 8 \\ \epsilon & 2 & \epsilon & \epsilon & 5 & 2 \\ \epsilon & 1 & 4 & 5 & \epsilon & 4 \\ \epsilon & \epsilon & 8 & 2 & 4 & \epsilon \end{pmatrix}$$

$$X(k+1) = A \otimes X(k) \quad \text{where} \quad \oplus \equiv \min$$

Shortest path problem is max-plus linear!

3. Modeling — Discrete event system and its dual description





Dater equations:

$$x_1(k) = 1x_1(k-2) \oplus 1x_2(k-1) \oplus 1u(k)$$

$$x_2(k) = 1x_1(k-1) \oplus 2u(k)$$

$$y(k) = x_1(k) \oplus x_2(k)$$

Counter equations:

$$x_1(t) = 2x_1(t-1) \oplus 1x_2(t-1) \oplus u(t-1)$$

$$x_2(t) = 1x_1(t-1) \oplus u(t-2)$$

$$y(t) = x_1(t) \oplus x_2(t)$$

Petri net is also max-plus linear!

3. Modeling — TCP network

F.Baccelli, D.Hong: TCP is Max-Plus Linear, ACM-SIGCOMM, 2000.

$$x(k+1) = A \otimes x(k) \quad x(k) : \text{dater vector}$$

- **TCP Tahoe and Reno are max-plus linear.**
- **Analytical evaluation of throughput.**
- **Analytical detection of losses and timeouts.**
- **Fractal scaling of TCP is also reported in later paper.**

Thus, Max-plus algebra is efficient for TCP traffic analysis.

4. System Theory — Conventional linear system theory

Conventional linear system

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k), \qquad \boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k)$$

The output $\boldsymbol{y}(k)$ is the convolution of impulse response and the input $\boldsymbol{u}(k)$.

$$\boldsymbol{y}(k) = \sum_{i=-\infty}^{k} \boldsymbol{h}(k-i)\boldsymbol{u}(i), \quad \boldsymbol{h}(k) : \text{impulse response}$$

The z-transform $X(z) \equiv \sum_{i=0}^{\infty} x(i)z^{-i}$ is convenient because we obtain the following representation:

$$\boldsymbol{Y}(z) = \boldsymbol{H}(z)\boldsymbol{U}(z)$$

The convolution becomes multiplication.

Linear system in max-plus algebra

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) \oplus \boldsymbol{B}\boldsymbol{u}(k), \qquad \boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k)$$

The output $\boldsymbol{y}(k)$ is the convolution of impulse response and the input $\boldsymbol{u}(k)$.

$$\boldsymbol{x}(k) = \bigoplus_{i=-\infty}^{k} \boldsymbol{h}(k-i) \otimes \boldsymbol{u}(i), \quad \boldsymbol{h}(k) : \text{impulse response}$$

The γ-transform $X(\gamma) \equiv \bigoplus_{i=0}^{\infty} x(i)\gamma^{-i}$ is convenient because we obtain the following representation:

$$\boldsymbol{Y}(\gamma) = \boldsymbol{H}(\gamma) \otimes \boldsymbol{U}(\gamma) = \boldsymbol{H}(\gamma) + \boldsymbol{U}(\gamma)$$

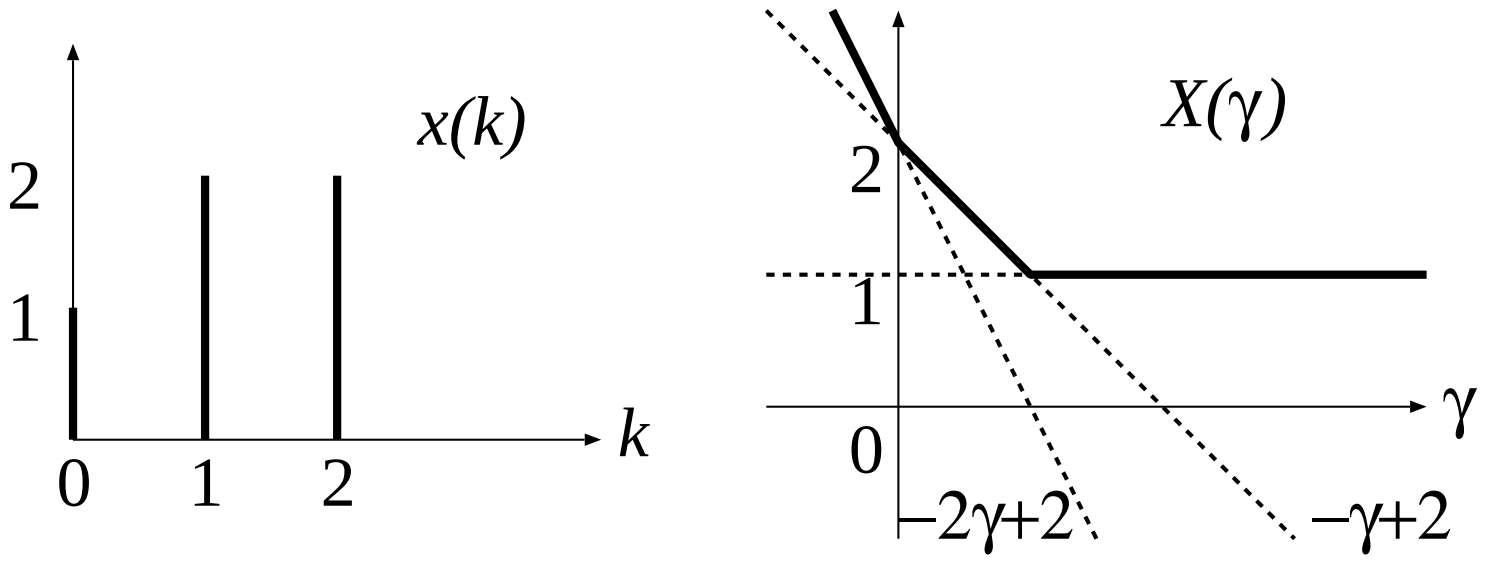
The convolution becomes multiplication$\otimes$.

4. System Theory — Example of γ-transform

$$X(\gamma) \equiv \bigoplus_k x(k)\gamma^{-k} = 1 \oplus 2\gamma^{-1} \oplus 2\gamma^{-2}$$

$$= \max(1,\ -\gamma + 2,\ -2\gamma + 2)$$

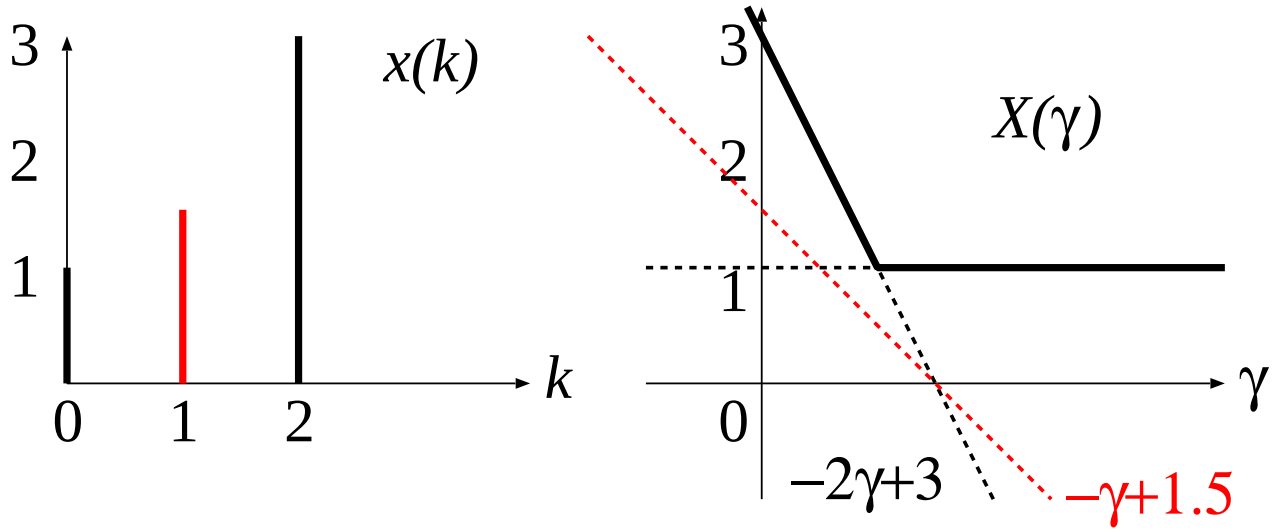
$$= \max_k(-k\gamma + x(k)) \quad : \text{Legendre transform!}$$



The γ-transform is equal to Legendre transform

4. System Theory — Example of γ-transform in nonconcave case

$$
\begin{aligned}
X(\gamma) &= 1 \oplus 1.5\gamma^{-1} \oplus 3\gamma^{-2} \\
&= \max(1,\ -\gamma + 1.5,\ -2\gamma + 3)
\end{aligned}
$$



The $x(1)$ is neglected because the function is non-concave and non-convex. In order to remove this drawback, we extend the Legendre (γ) transform.

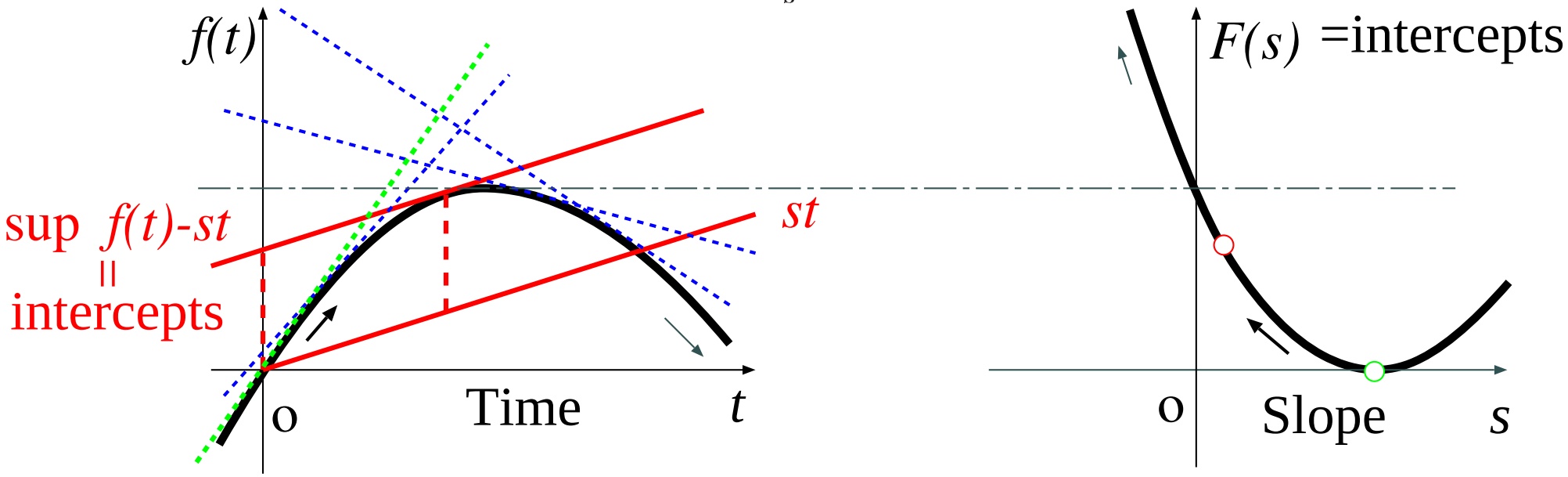
5. Legendre transform — Definition

Legendre transform

$$\mathcal{L}[f(t)](s) = \sup_t (f(t) - st)$$

Inverse Legendre transform

$$\mathcal{L}^{-1}[F(s)](t) = \inf_s (F(s) + st)$$



Legendre transform is the description with set of tangent.

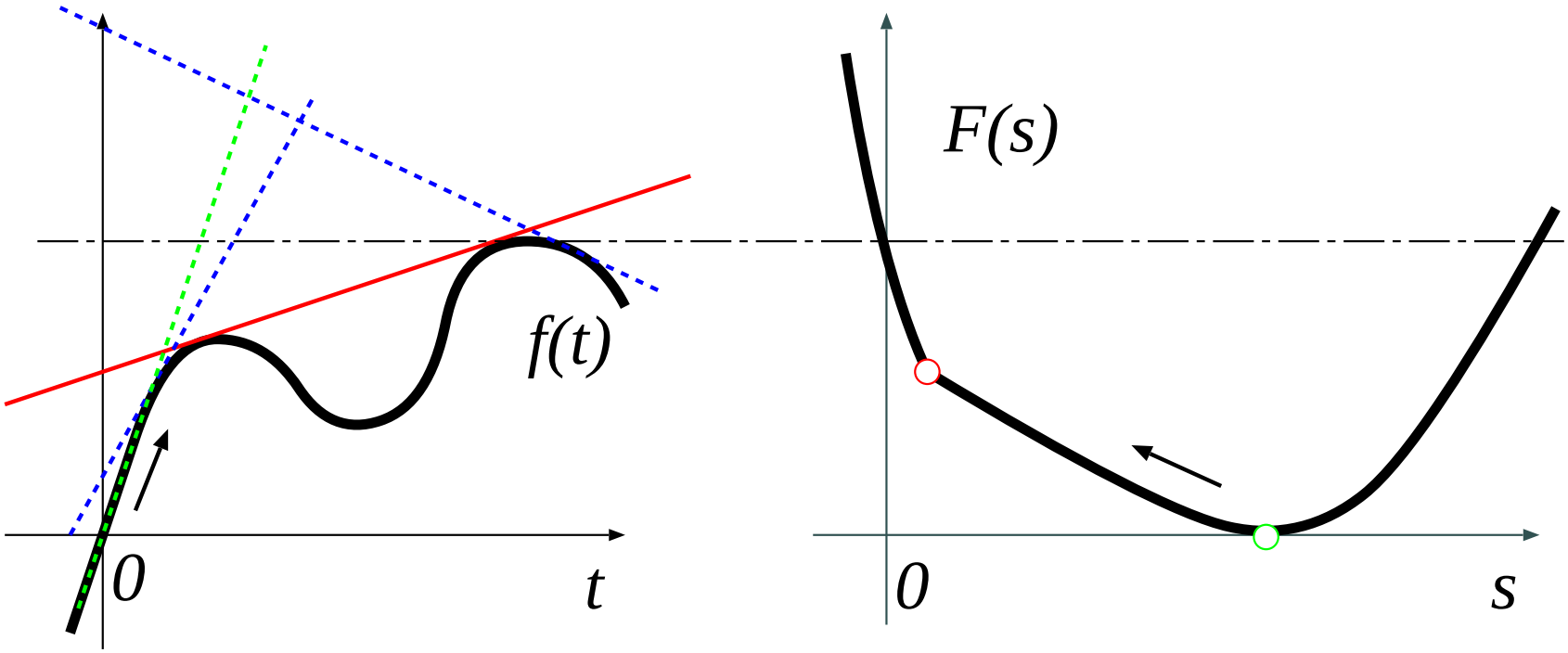
5. Legendre transform — Property of the Legendre transform

$$
\begin{aligned}
a + f(t) &\Leftrightarrow a + \mathcal{L}[f(t)](s) \\
f(t-a) &\Leftrightarrow \mathcal{L}[f(t)](s) - sa \\
f(t) + at &\Leftrightarrow \mathcal{L}[f(t)](s-a) \\
f(at) &\Leftrightarrow \mathcal{L}[f(t)](s/a) \\
af(t) &\Leftrightarrow a\mathcal{L}[f(t)](s/a) \\
\sup_\tau(f(\tau) + g(t-\tau)) &\Leftrightarrow \mathcal{L}[f(t)](s) + \mathcal{L}[g(t)](s) \\
f(t) + g(t) &\Leftrightarrow \sup_\sigma(\mathcal{L}[f(t)](\sigma) + \mathcal{L}[g(t)](s-\sigma))
\end{aligned}
$$

The above relations are satisfied for convex or concave functions.

5. Legendre transform — Non-convex or non-concave function

In the case of non-convex or non-concave function, the Legendre transform is not suitable for the description of function, because some part of function is neglected.

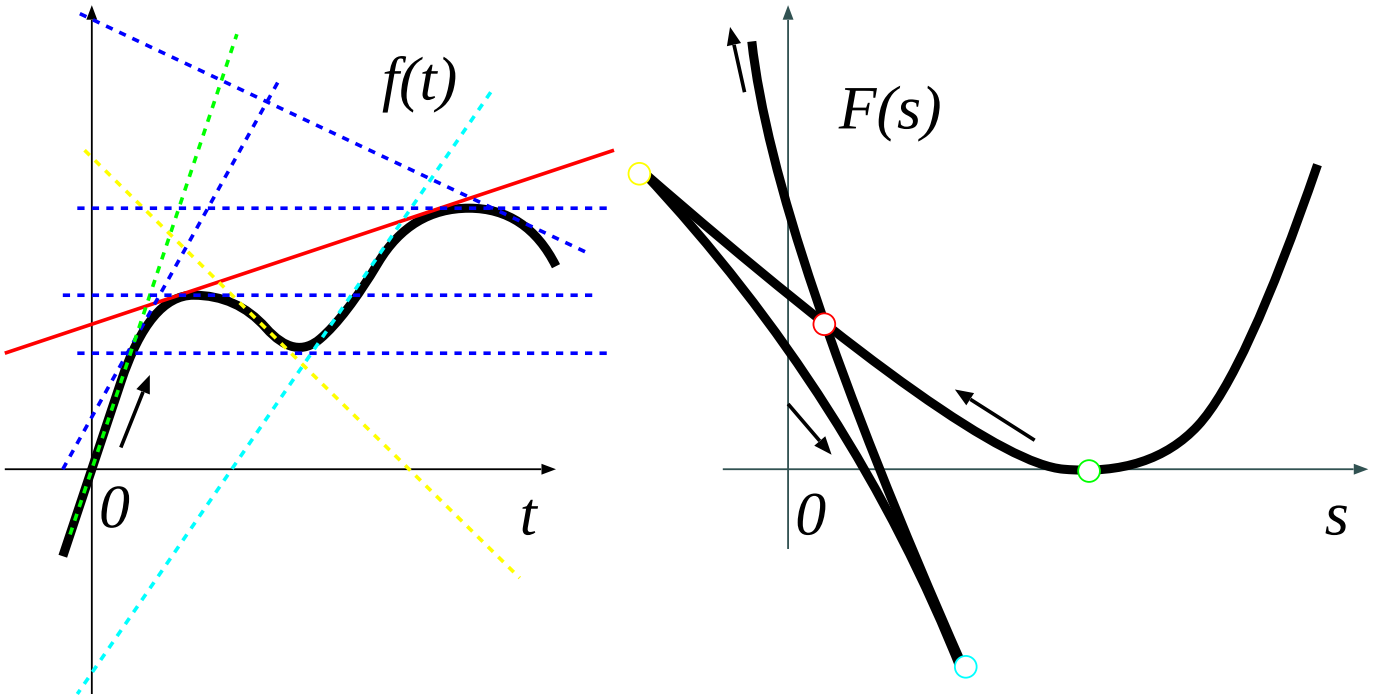


Extended Legendre transform

$$\mathcal{L}[f(t)](s) = \{f(t^*) - st^* \mid f'(t^*) = s\}$$

Extended inverse Legendre transform

$$\mathcal{L}^{-1}[F(s)](t) = \{F(s^*) + ts^* \mid F'(s^*) = t\}$$



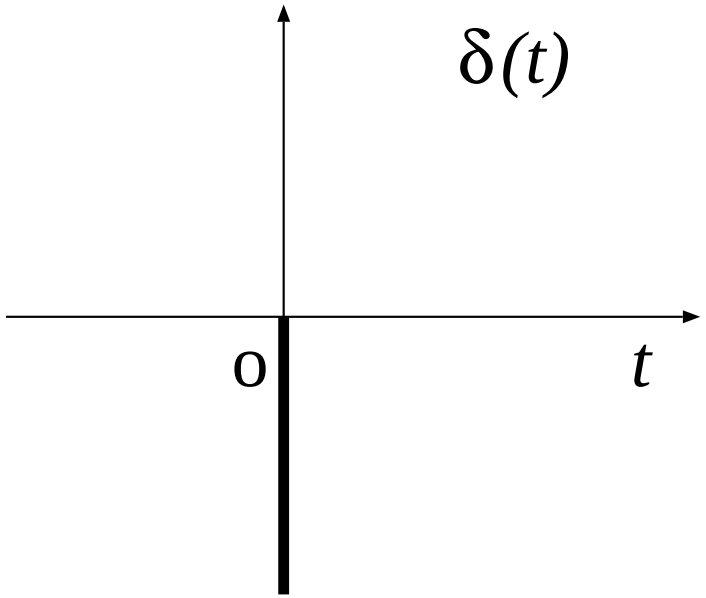
The complete set of tangent line is considered.

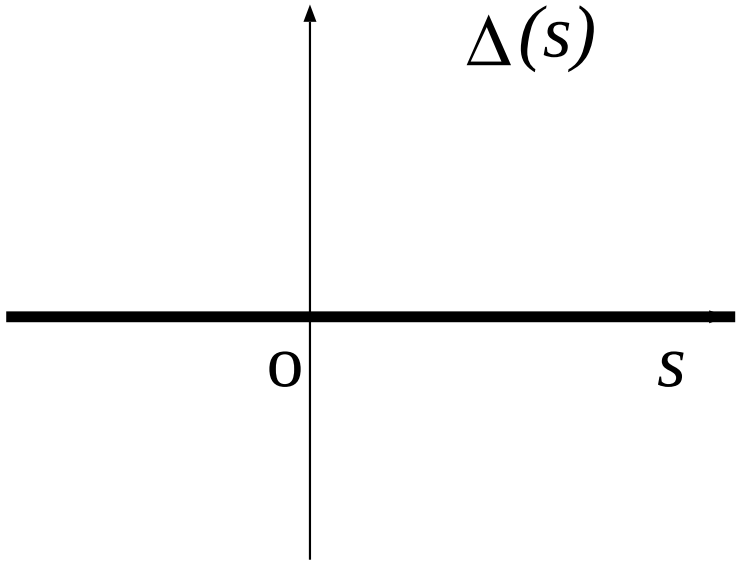
5. Legendre transform — Property of the Legendre transform

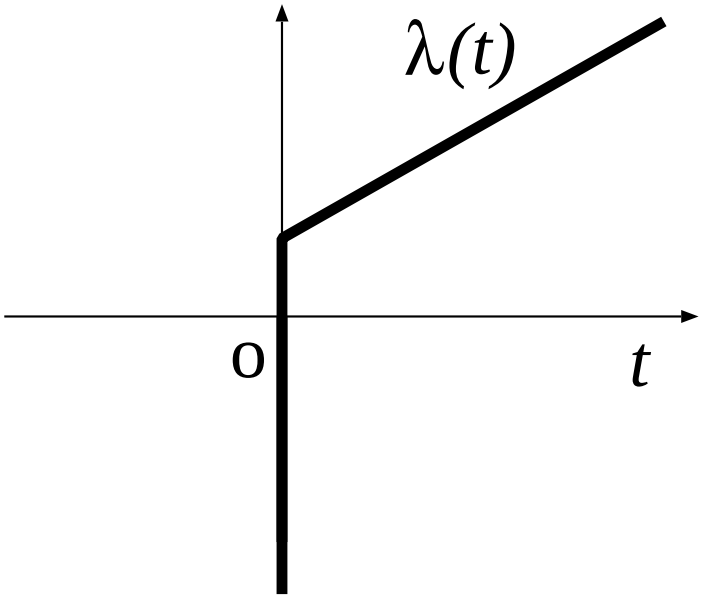
$$
\begin{aligned}
a + f(t) &\Leftrightarrow a + \mathcal{L}[f(t)](s) \\
f(t-a) &\Leftrightarrow \mathcal{L}[f(t)](s) - sa \\
f(t) + at &\Leftrightarrow \mathcal{L}[f(t)](s-a) \\
f(at) &\Leftrightarrow \mathcal{L}[f(t)](s/a) \\
af(t) &\Leftrightarrow a\mathcal{L}[f(t)](s/a) \\
\sup_\tau(f(\tau) + g(t-\tau)) &\Leftrightarrow \mathcal{L}[f(t)](s) + \mathcal{L}[g(t)](s) \\
f(t) + g(t) &\Leftrightarrow \sup_\sigma(\mathcal{L}[f(t)](\sigma) + \mathcal{L}[g(t)](s-\sigma))
\end{aligned}
$$

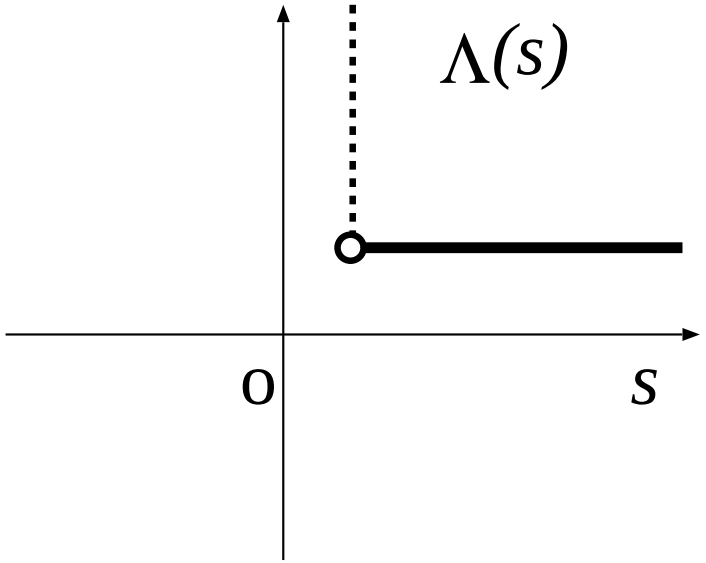
The above relations are satisfied, even if functions are non-convex and non-concave.

5. Legendre transform — Examples of Legendre transform

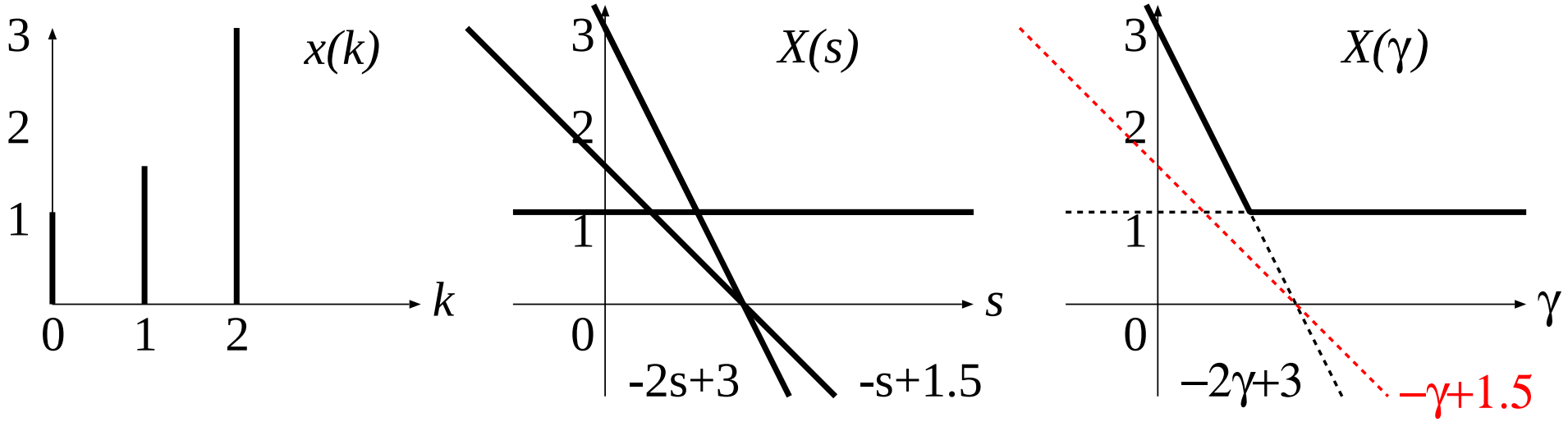






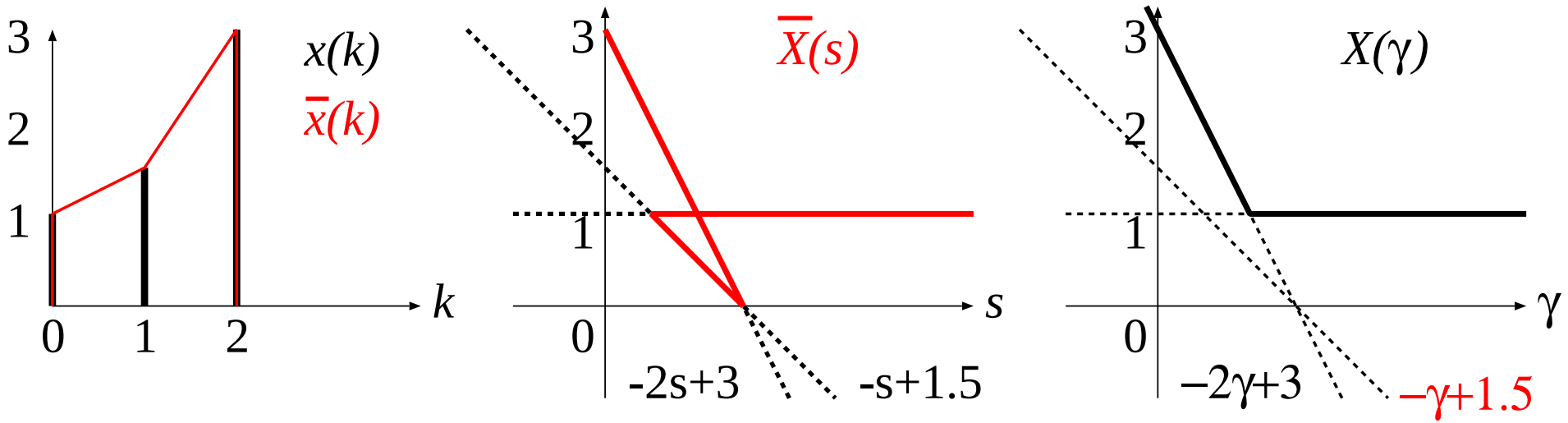


5. Legendre transform — Non-convex and non-concave case



The drawback is eliminated. However, the function is very complicated.

5. Legendre transform — Piecewise linear interpolation



Interpolation is effective for simplification.

For time series analysis, we apply the extended Legendre transform with interpolation instead of γ transform.

6. Application — Network analysis with Legendre transform

We consider the following max-plus linear network system.

$$y(k) = \bigoplus_{i=-\infty}^{k} h(k-i) \otimes u(i),$$

$u(k)$: input trace
$h(k)$: impulse response
$y(k)$: output trace

1. **Calculate interpolation $\bar{h}(t)$ and $\bar{u}(t)$.**
2. **Apply the extended Legendre transform to $\bar{h}(t)$ and $\bar{u}(t)$ and obtain $\bar{H}(s)$ and $\bar{U}(s)$.**
3. **Calculate $\bar{Y}(s) = \bar{H}(s) + \bar{U}(s)$ with addition.**
4. **Obtain $\bar{y}(k)$ using the extended inverse Legendre transform.**

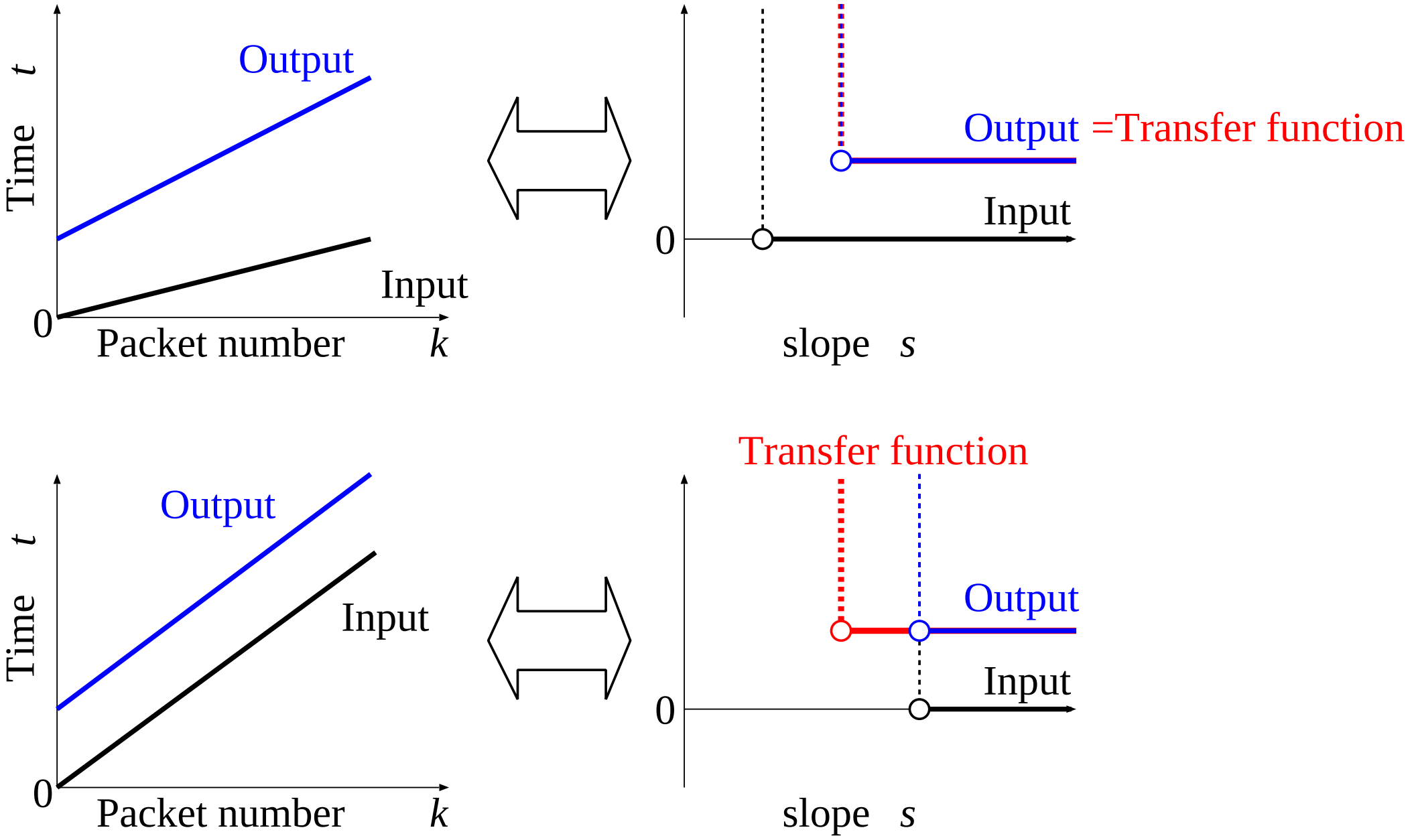
The fast Legendre transform is available.

6. Application — Identification of network system

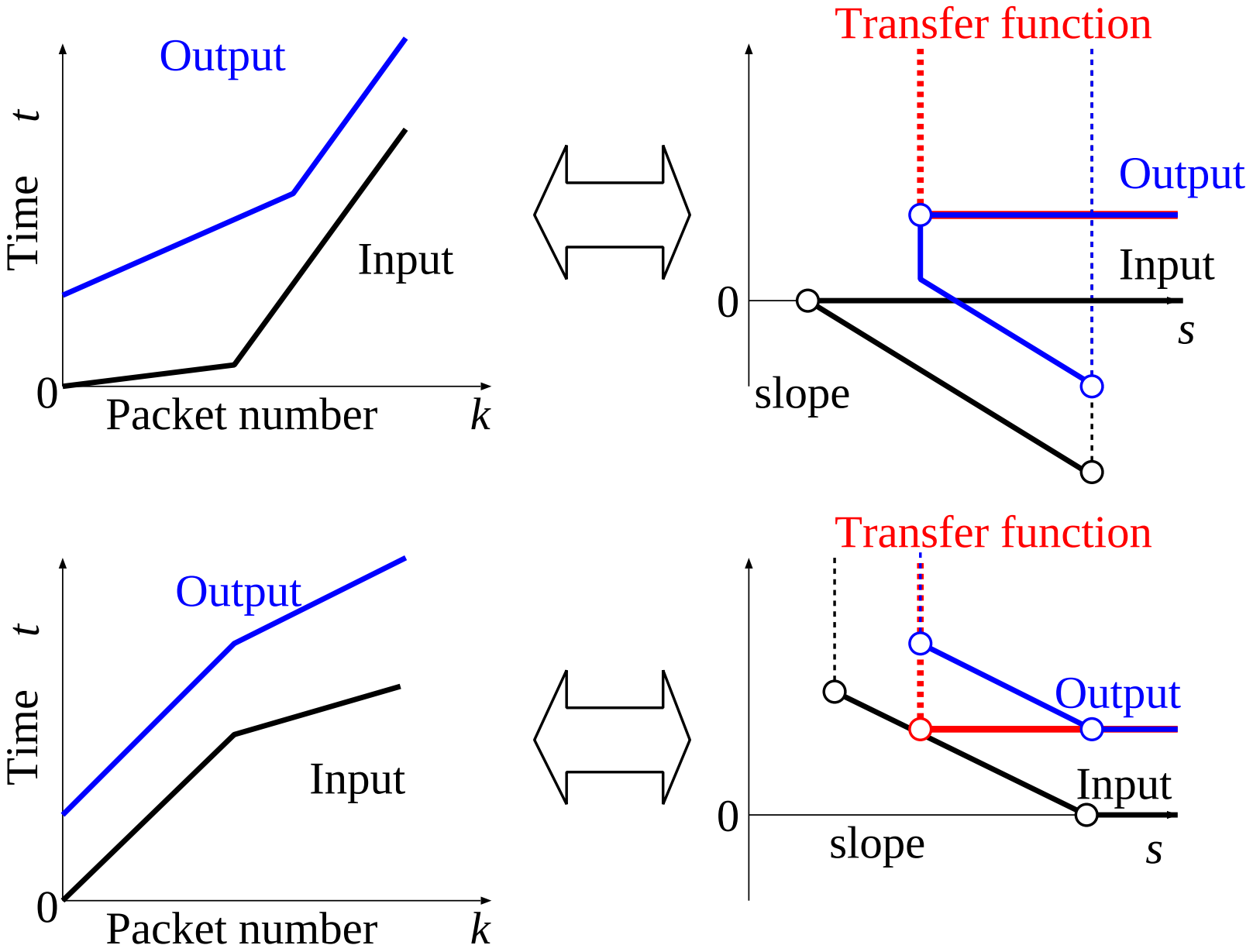
For traffic analysis, we need impulse response of transfer function. In order to obtain the transfer function, we use the system identification with deconvolution.

1. Obtain the input trace $u(k)$ and output trace $y(k)$
2. Calculate interpolation $\bar{y}(t)$ and $\bar{u}(t)$.
3. Apply the extended Legendre transform to $\bar{u}(t)$ and $\bar{y}(t)$ and obtain $\bar{U}(s)$ and $\bar{Y}(s)$.
4. Calculate the transfer function $\bar{H}(s) = \bar{Y}(s) - \bar{U}(s)$. This is the deconvoloution process.
5. Obtain impulse response $\bar{h}(t)$ using the extended inverse Legendre transform.

6. Application — Transfer function of Bottleneck router



6. Application — Representation of traffic change



7. Conclusion

- TCP is linear in max-plus algebra.
- The output is given by the convolution of input and transfer function.
- The extended Legendre transform in max-plus algebra corresponds to Fourier transform and z-transform in linear system theory.
- The deconvolution is given by the difference of the transformed input and output.
- Thus, we can identify the transfer function of network system using the extended Legendre transform.

8. Future work

- We have to confirm the effectiveness using simulation.
- We can try to apply this method to any protocols and any networks.

References

[1] **F.Baccelli, G.Cohen, G.J.Olsder, and J.P.Quadrant:** *Synchronization and Linearity*, **Wiley, 1992.**

[2] **F.Baccelli and D.Hong : "TCP is Max-Plus Linear," Proc. of ACM-SIGCOMM'00, Vol.30, No.4, pp.219–230, 2000.**

[3] **F.Baccelli and D. Hong : "AIMD, Fairness and Fractal Scaling of TCP Traffic," Proc. of IEEE INFOCOM, 2002.**

[4] **V.P.Maslov and S.N.Samborskii:** *Idempotent Analysis*, **AMS, 1992.**

[5] **L.Dorst and R. van den Boomgaard:"Morphological Signal Processing and the Slope Transform," Signal Processing, Vol.38, pp.79–98, 1994.**

[6] **P.Maragos:" Slope Transforms:Theory and Application to Nonlinear Signal Processing," IEEE Trans. Signal Processing, Vol.43, No.4, pp.864–877, 1995.**

10. Appendix — Why Legendre transform?

Conventional Fourier transform

$$F(s) = \int f(t)e^{-st}\,\mathrm{d}t \tag{1}$$

e^{-st} is based on characters of the group:

$$f(x+y) = f(x)f(y).$$

Max-plus algebra analog is

$$f(x+y) = f(x) + f(y).$$

Max-plus characters of that group are $-s \cdot x = \log e^{-st}$
Fourier transform in max-plus algebra is

$$F(s) = \max(f(t) - s \cdot t) \tag{2}$$

That is Legendre transform!