Traffic Analysis and System Identification with Legendre Transform

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Objective

- Traffic analysis in communication network
- Identification of communication network system

Network system is nonlinear.
$\Longrightarrow$ The analysis is very complicated.

Tools

- Max-plus algebra
- Legendre transform

1. Introduction — Linear description of nonlinear system

Max-plus algebra is defined by

$$\text{addition} :\oplus \equiv \max \qquad \text{and} \qquad \text{multiplication} :\otimes \equiv + \quad .$$

We can obtain linear description of nonlinear systems;

- **Discrete event systems**
- **Dynamic programming**
- **Markov decision processes (Viterbi algorithm)**
- **Morphological operations**

etc

TCP network is also max-plus linear.

1. Introduction — Our project

For the system described with max-plus algebra, Legendre transform is a powerful tool.

Legendre transform in max-plus algebra
$\Longrightarrow$ Fourier transform and z-transform in linear system

Legendre transform gives

- **Time series analysis**
  $\Rightarrow$ Traffic analysis
- **Deconvolution**
  $\Rightarrow$ System identification

Our objective is to clarify complicated traffic and to identify complicated network system with Legendre transform.

2. Max-plus algebra — Definition

- **Addition** $\oplus$

$$\oplus \equiv \max$$

- **Multiplication** $\otimes$

$$\otimes \equiv +$$

Example

$$\begin{aligned} 1 \oplus 2 &= \max(1,2) = 2 \\ 2 \otimes 3 &= 2 + 3 = 5 \end{aligned}$$

2. Max-plus algebra — Linear system

Difference equations

$$\boldsymbol{x}(t+1) = \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

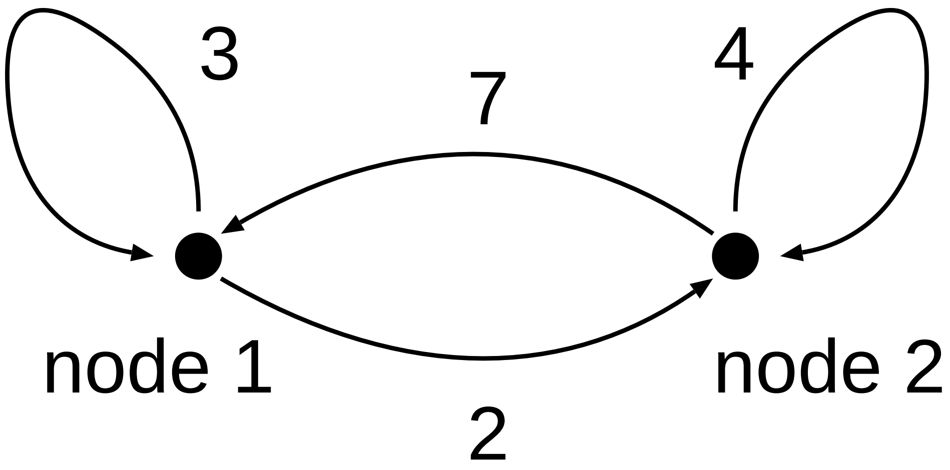
Conventional difference equations

$$\boldsymbol{x}(1) = \begin{pmatrix} 3 \times 1 + 7 \times 0 \\ 2 \times 1 + 4 \times 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}(2) = \begin{pmatrix} 23 \\ 14 \end{pmatrix}, \cdots$$

Max-plus difference equations

$$\boldsymbol{x}(1) = \begin{pmatrix} \max(3+1, 7+0) \\ \max(2+1, 4+0) \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}, \quad \boldsymbol{x}(2) = \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \cdots$$

2. Max-plus algebra — Simple example



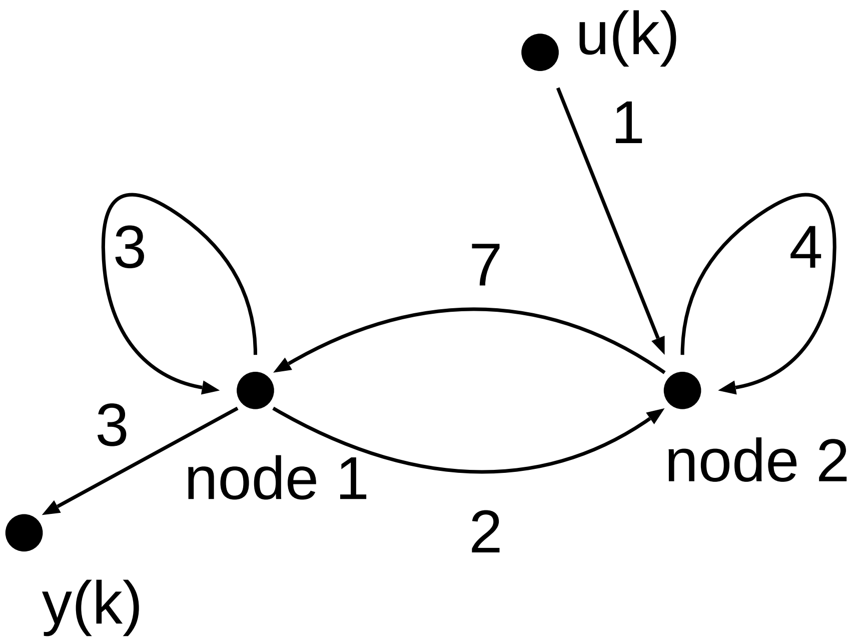
$x_i(k)$ **is the earliest epoch when node** i **becomes active for the** k**-th time.**

A_{ij} **is the traveling time from node** j **to node** i**.**

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} \max(3 + x_1(k), 7 + x_2(k)) \\ \max(2 + x_1(k), 4 + x_2(k)) \end{pmatrix} = \boldsymbol{A}\boldsymbol{x}(k)$$

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}(1) = \begin{pmatrix} 7 \\ 4 \end{pmatrix}, \quad \boldsymbol{x}(2) = \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \cdots$$

2. Max-plus algebra — Example with input and output



$$
\begin{aligned}
\boldsymbol{x}(k+1) &= \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \boldsymbol{x}(k) \oplus \begin{pmatrix} \epsilon \\ 1 \end{pmatrix} u(k) \\
y(k) &= \begin{pmatrix} 3 & \epsilon \end{pmatrix} \boldsymbol{x}(k)
\end{aligned}
$$

$$\epsilon \equiv -\infty$$

2. Max-plus algebra — Definition of neutral element

Neutral element with respect to maximization

$$\epsilon \equiv -\infty, \qquad a \oplus \epsilon = a$$

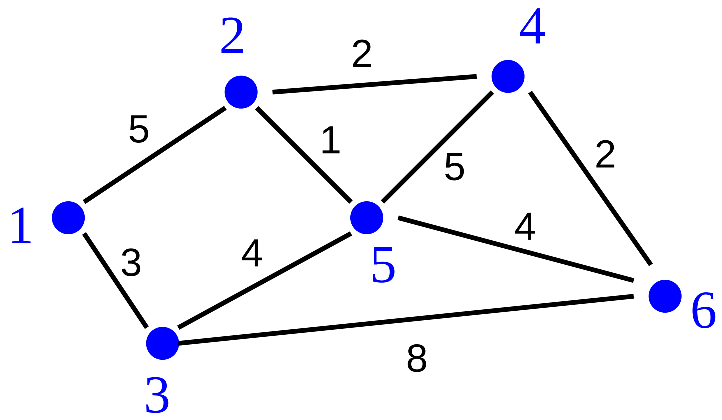
Neutral element with respect to addition

$$e \equiv 0, \qquad a \otimes e = a$$

Relation between conventional and max-plus algebra

$$\begin{aligned} -\infty &= \log 0 \\ 0 &= \log 1 \end{aligned}$$

3. Modeling — Shortest path problem



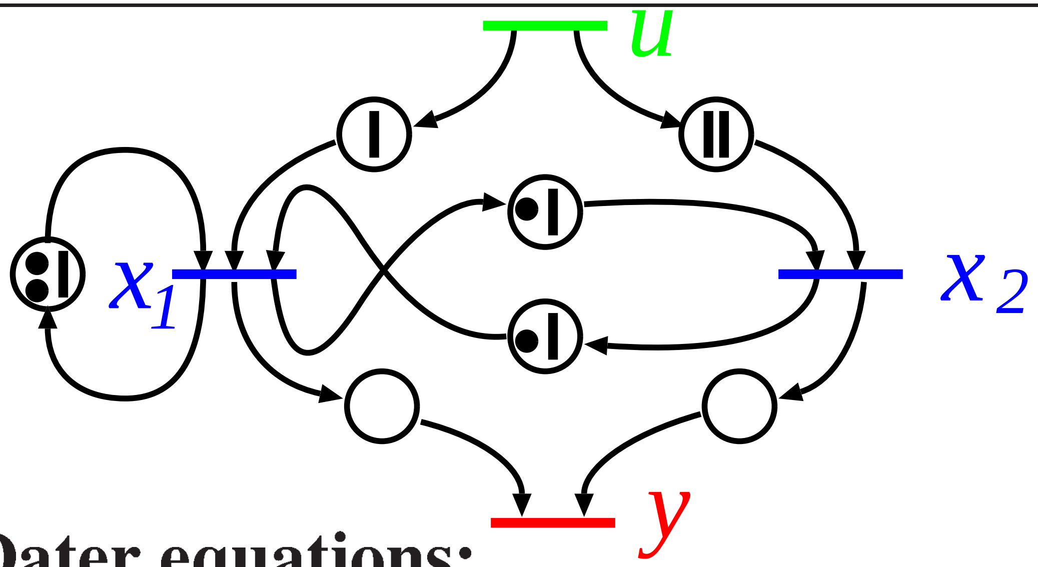
$X_{ij}(k)$ represents the shortest path from j to i with k-path

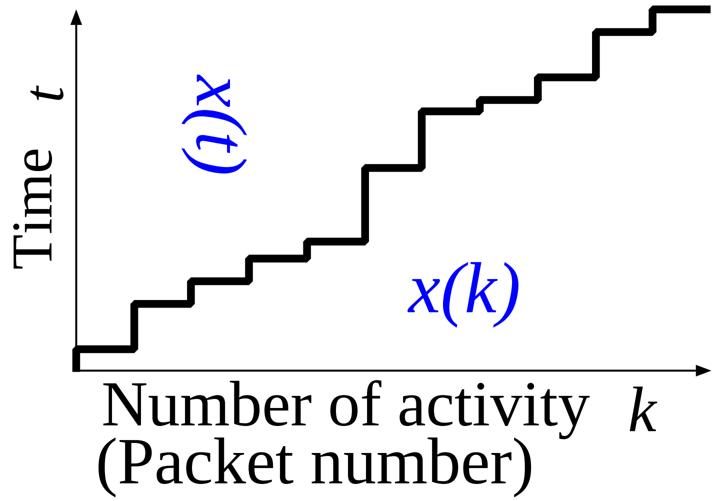
$$X_{ij}(k+1) \quad = \quad \min_{l=1,\ldots,6} (A_{il} + X_{lj}(k)), \qquad \boldsymbol{A} = \begin{pmatrix} \epsilon & 5 & 3 & \epsilon & \epsilon & \epsilon \\ 5 & \epsilon & \epsilon & 2 & 1 & \epsilon \\ 3 & \epsilon & \epsilon & \epsilon & 4 & 8 \\ \epsilon & 2 & \epsilon & \epsilon & 5 & 2 \\ \epsilon & 1 & 4 & 5 & \epsilon & 4 \\ \epsilon & \epsilon & 8 & 2 & 4 & \epsilon \end{pmatrix}$$

$$\boldsymbol{X}(k+1) \quad = \quad \boldsymbol{A} \otimes \boldsymbol{X}(k) \qquad \text{where} \quad \oplus \equiv \min$$

Shortest path problem is max-plus linear!

3. Modeling — Discrete event system and its dual description





Dater equations:

$$
\begin{aligned}
x_1(k) &= 1x_1(k-2) \oplus 1x_2(k-1) \oplus 1u(k) \\
x_2(k) &= 1x_1(k-1) \oplus 2u(k) \\
y(k) &= x_1(k) \oplus x_2(k)
\end{aligned}
$$

Counter equations:

$$
\begin{aligned}
x_1(t) &= 2x_1(t-1) \oplus 1x_2(t-1) \oplus u(t-1) \\
x_2(t) &= 1x_1(t-1) \oplus u(t-2) \\
y(t) &= x_1(t) \oplus x_2(t)
\end{aligned}
$$

Petri net is also max-plus linear!

3. Modeling — TCP network

F.Baccelli, D.Hong: TCP is Max-Plus Linear, ACM-SIGCOMM, 2000.

$$\boldsymbol{x}(k+1) = A \otimes \boldsymbol{x}(k) \qquad x(k) : \text{dater vector}$$

- **TCP Tahoe and Reno are max-plus linear.**
- **Analytical evaluation of throughput.**
- **Analytical detection of losses and timeouts.**
- **Fractal scaling of TCP is also reported in later paper.**

Thus, Max-plus algebra is efficient for TCP traffic analysis.

Conventional linear system

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k), \qquad \boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k)$$

The output $\boldsymbol{y}(k)$ is the convolution of impulse response and the input $\boldsymbol{u}(k)$.

$$\boldsymbol{y}(k) = \sum_{i=-\infty}^{k} \boldsymbol{h}(k-i)\boldsymbol{u}(i), \quad \boldsymbol{h}(k) : \text{impulse response}$$

The z-transform $X(z) \equiv \sum_{i=0}^{\infty} x(i)z^{-i}$ is convenient because we obtain the following representation:

$$\boldsymbol{Y}(z) = \boldsymbol{H}(z)\boldsymbol{U}(z)$$

The convolution becomes multiplication.

Linear system in max-plus algebra

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) \oplus \boldsymbol{B}\boldsymbol{u}(k), \qquad \boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k)$$

The output $\boldsymbol{y}(k)$ is the convolution of impulse response and the input $\boldsymbol{u}(k)$.

$$\boldsymbol{x}(k) = \bigoplus_{i=-\infty}^{k} \boldsymbol{h}(k-i) \otimes \boldsymbol{u}(i), \quad \boldsymbol{h}(k) : \text{impulse response}$$

The γ-transform $X(\gamma) \equiv \bigoplus_{i=0}^{\infty} x(i)\gamma^{-i}$ is convenient because we obtain the following representation:

$$Y(\gamma) = \boldsymbol{H}(\gamma) \otimes \boldsymbol{U}(\gamma) = \boldsymbol{H}(\gamma) + \boldsymbol{U}(\gamma)$$

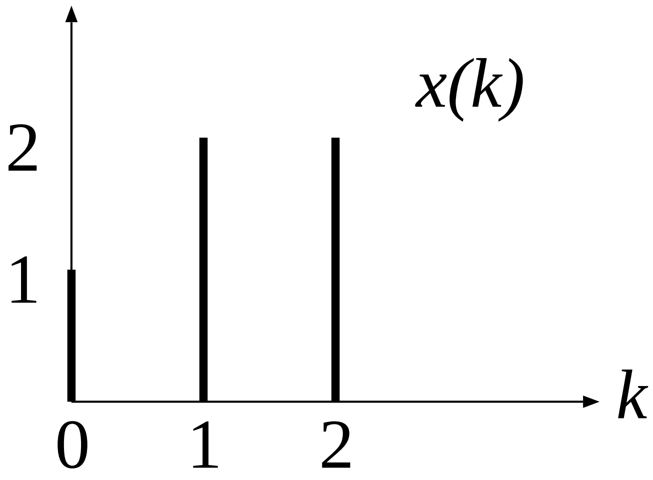
The convolution becomes multiplication$\otimes$.

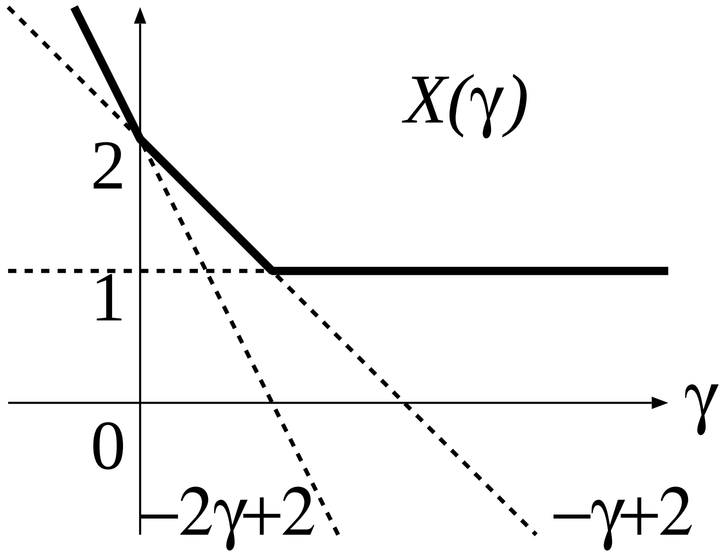
4. System Theory — Example of γ-transform

$$X(\gamma) \equiv \bigoplus_k x(k)\gamma^{-k} = 1 \oplus 2\gamma^{-1} \oplus 2\gamma^{-2}$$

$$= \max(1,\ -\gamma+2,\ -2\gamma+2)$$

$$= \max_k(-k\gamma + x(k))$$: Legendre transform!

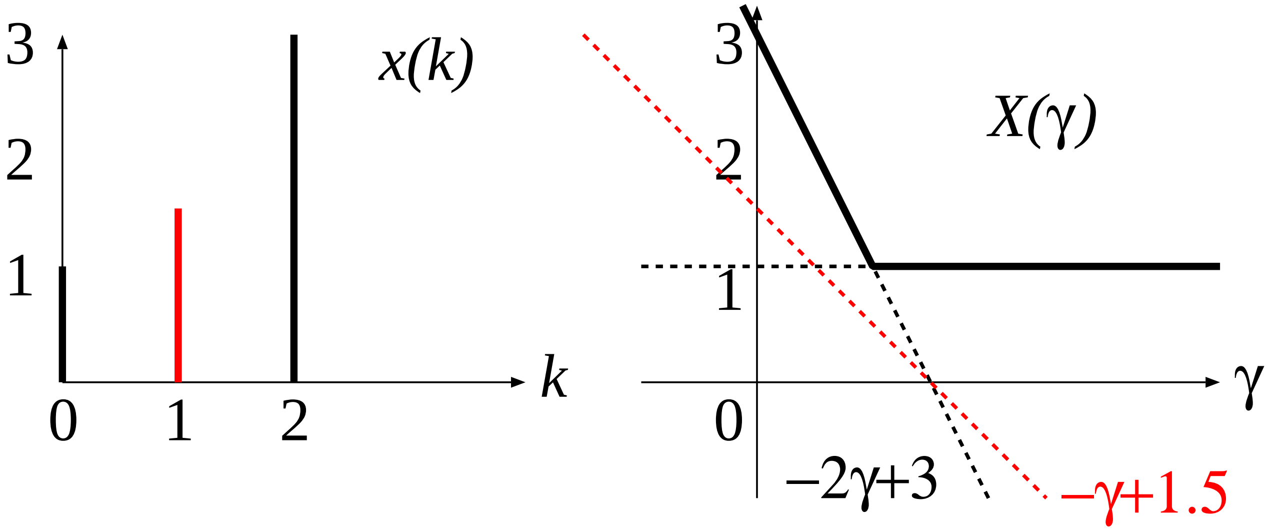




The γ-transform is equal to Legendre transform

4. System Theory — Example of γ-transform in nonconcave case

$$\begin{aligned} X(\gamma) &= 1 \oplus 1.5\gamma^{-1} \oplus 3\gamma^{-2} \\ &= \max(1,\ -\gamma + 1.5,\ -2\gamma + 3) \end{aligned}$$



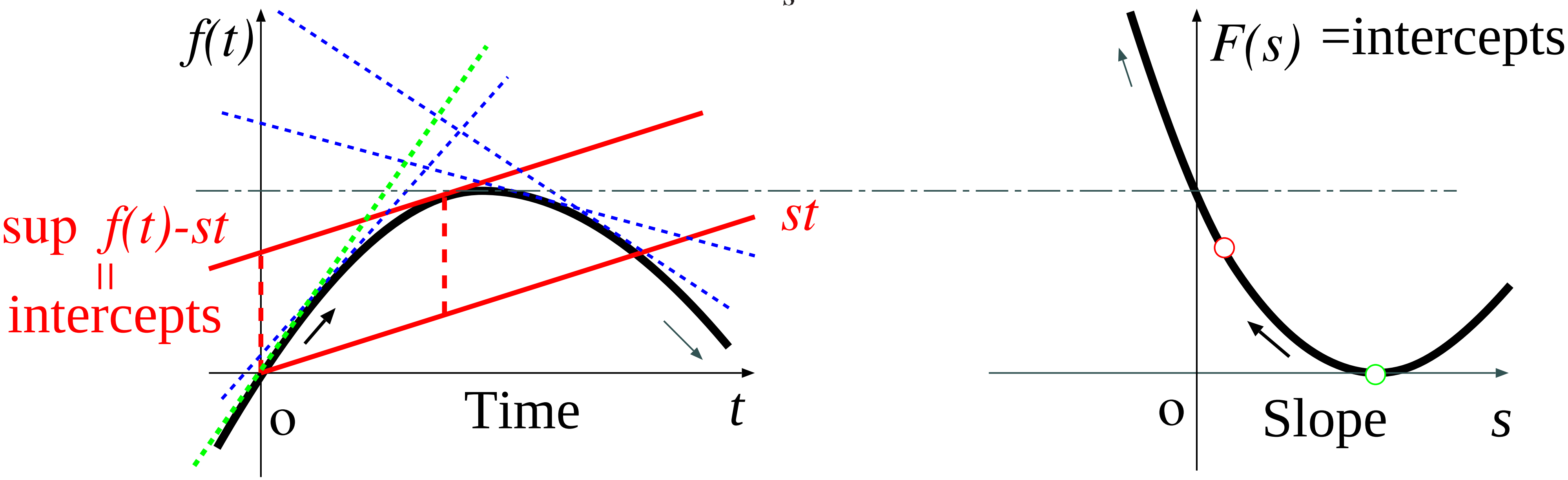
The $x(1)$ is neglected because the function is non-concave and non-convex. In order to remove this drawback, we extend the Legendre (γ) transform.

Legendre transform

$$\mathcal{L}[f(t)](s) = \sup_t (f(t) - st)$$

Inverse Legendre transform

$$\mathcal{L}^{-1}[F(s)](t) = \inf_s (F(s) + st)$$



Legendre transform is the description with set of tangent.

5. Legendre transform — Property of the Legendre transform

$$a + f(t) \quad \Leftrightarrow \quad a + \mathcal{L}[f(t)](s)$$

$$f(t-a) \quad \Leftrightarrow \quad \mathcal{L}[f(t)](s) - sa$$

$$f(t) + at \quad \Leftrightarrow \quad \mathcal{L}[f(t)](s-a)$$

$$f(at) \quad \Leftrightarrow \quad \mathcal{L}[f(t)](s/a)$$

$$af(t) \quad \Leftrightarrow \quad a\mathcal{L}[f(t)](s/a)$$

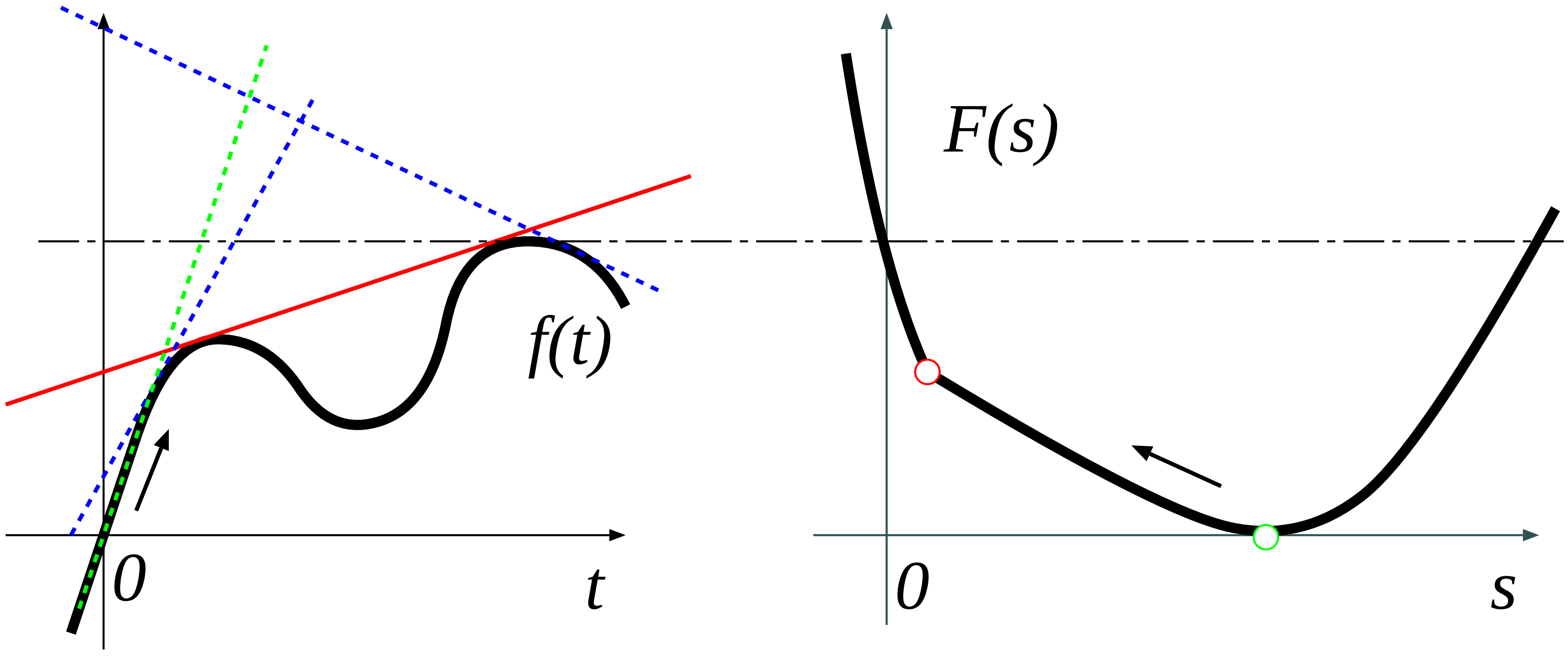
$$\sup_\tau(f(\tau) + g(t-\tau)) \quad \Leftrightarrow \quad \mathcal{L}[f(t)](s) + \mathcal{L}[g(t)](s)$$

$$f(t) + g(t) \quad \Leftrightarrow \quad \sup_\sigma(\mathcal{L}[f(t)](\sigma) + \mathcal{L}[g(t)](s-\sigma))$$

The above relations are satisfied for convex or concave functions.

5. Legendre transform — Non-convex or non-concave function

In the case of non-convex or non-concave function, the Legendre transform is not suitable for the description of function, because **some part of function is neglected**.

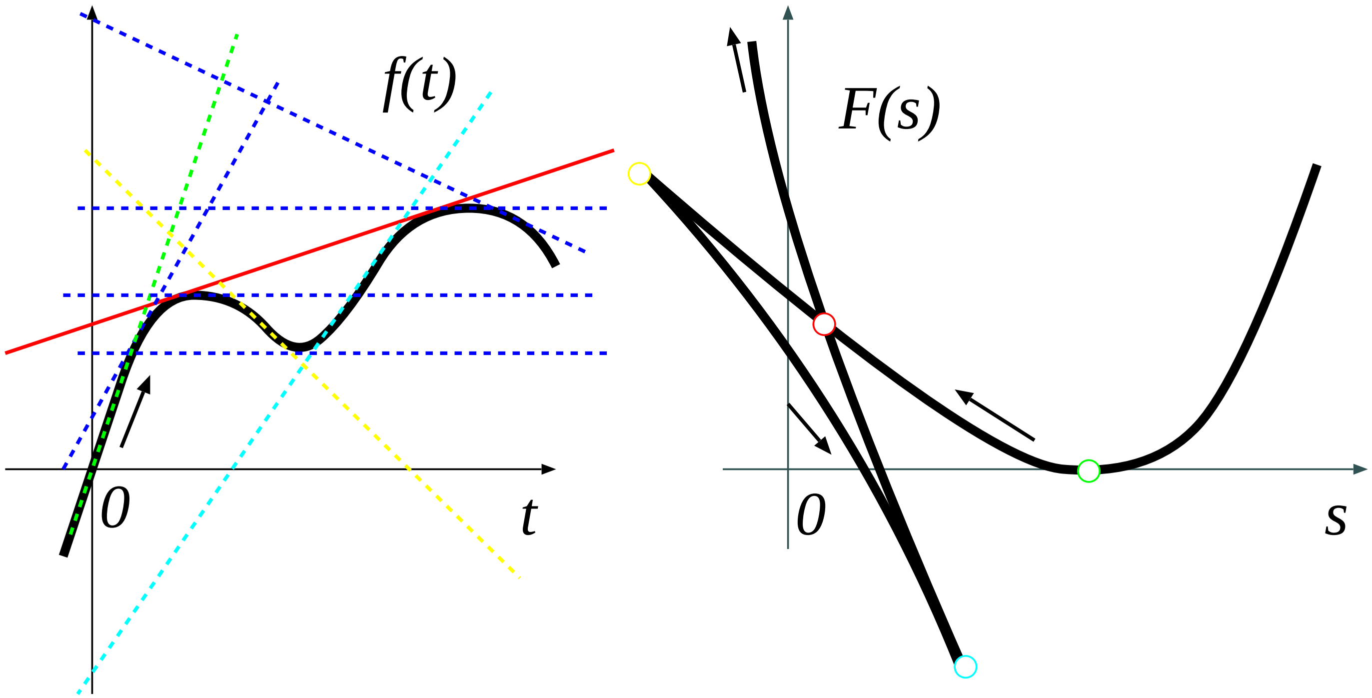


Extended Legendre transform

$$\mathcal{L}[f(t)](s) = \{f(t^*) - st^* \mid f'(t^*) = s\}$$

Extended inverse Legendre transform

$$\mathcal{L}^{-1}[F(s)](t) = \{F(s^*) + ts^* \mid F'(s^*) = t\}$$



The complete set of tangent line is considered.

5. Legendre transform — Property of the Legendre transform

$$a + f(t) \quad \Leftrightarrow \quad a + \mathcal{L}[f(t)](s)$$

$$f(t-a) \quad \Leftrightarrow \quad \mathcal{L}[f(t)](s) - sa$$

$$f(t) + at \quad \Leftrightarrow \quad \mathcal{L}[f(t)](s-a)$$

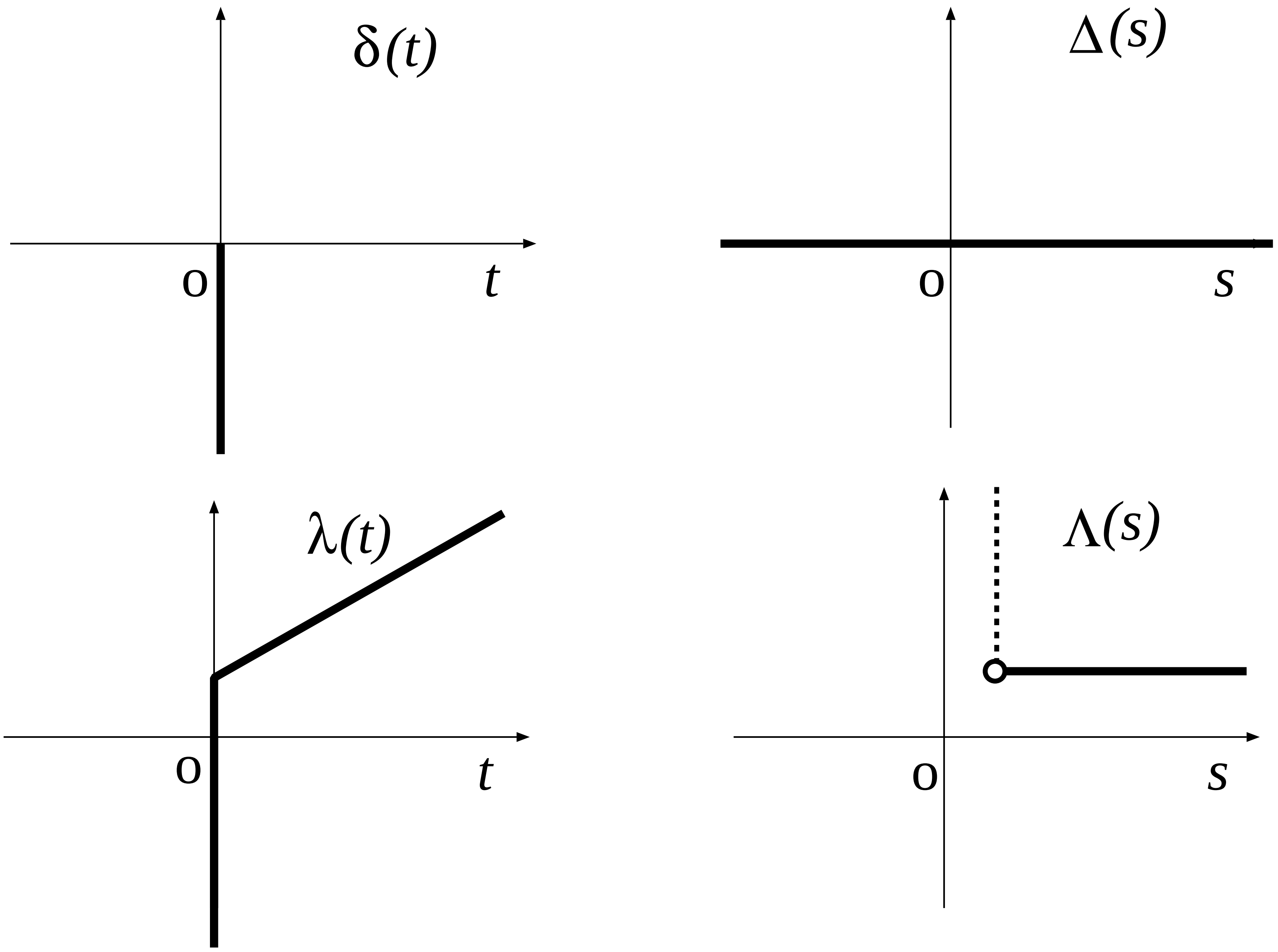
$$f(at) \quad \Leftrightarrow \quad \mathcal{L}[f(t)](s/a)$$

$$af(t) \quad \Leftrightarrow \quad a\mathcal{L}[f(t)](s/a)$$

$$\sup_\tau(f(\tau) + g(t-\tau)) \quad \Leftrightarrow \quad \mathcal{L}[f(t)](s) + \mathcal{L}[g(t)](s)$$

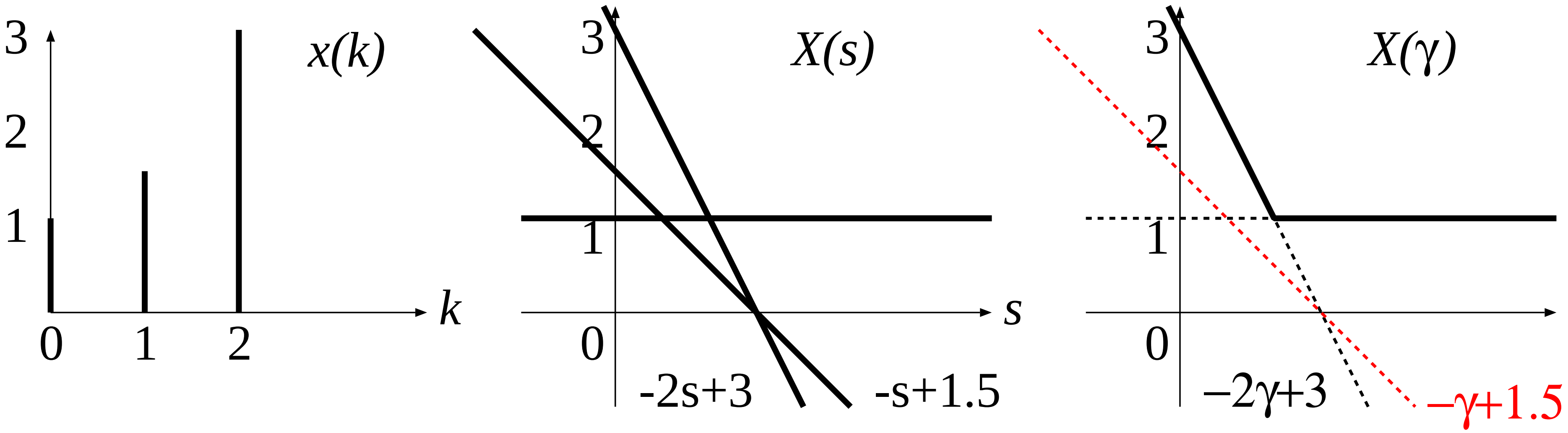
$$f(t) + g(t) \quad \Leftrightarrow \quad \sup_\sigma(\mathcal{L}[f(t)](\sigma) + \mathcal{L}[g(t)](s-\sigma))$$

The above relations are satisfied, even if functions are non-convex and non-concave.



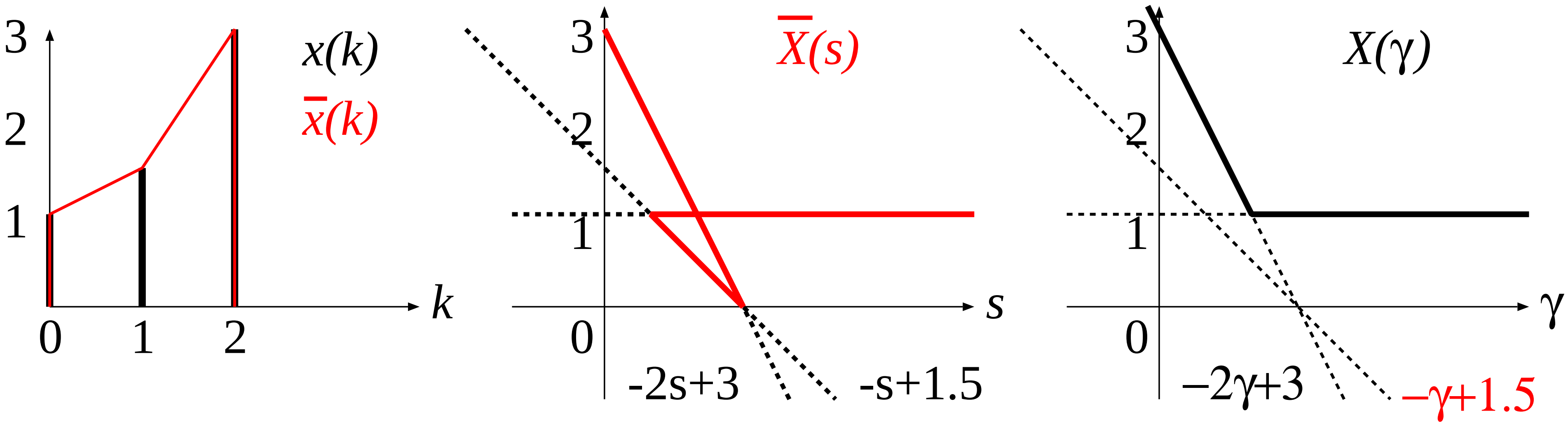
δ(t)
O
t
Δ(s)
O
s
λ(t)
O
t
Λ(s)
O
s

5. Legendre transform — Non-convex and non-concave case



The drawback is eliminated. However, the function is very complicated.

5. Legendre transform — Piecewise linear interpolation



Interpolation is effective for simplification.

For time series analysis, we apply the extended Legendre transform with interpolation instead of γ transform.

6. Application — Network analysis with Legendre transform

We consider the following max-plus linear network system.

$$y(k) = \bigoplus_{i=-\infty}^{k} h(k-i) \otimes u(i),$$

$u(k)$: input trace
$h(k)$: impulse response
$y(k)$: output trace

1. **Calculate interpolation $\bar{h}(t)$ and $\bar{u}(t)$.**
2. **Apply the extended Legendre transform to $\bar{h}(t)$ and $\bar{u}(t)$ and obtain $\bar{H}(s)$ and $\bar{U}(s)$.**
3. **Calculate $\bar{Y}(s) = \bar{H}(s) + \bar{U}(s)$ with addition.**
4. **Obtain $\bar{y}(k)$ using the extended inverse Legendre transform.**

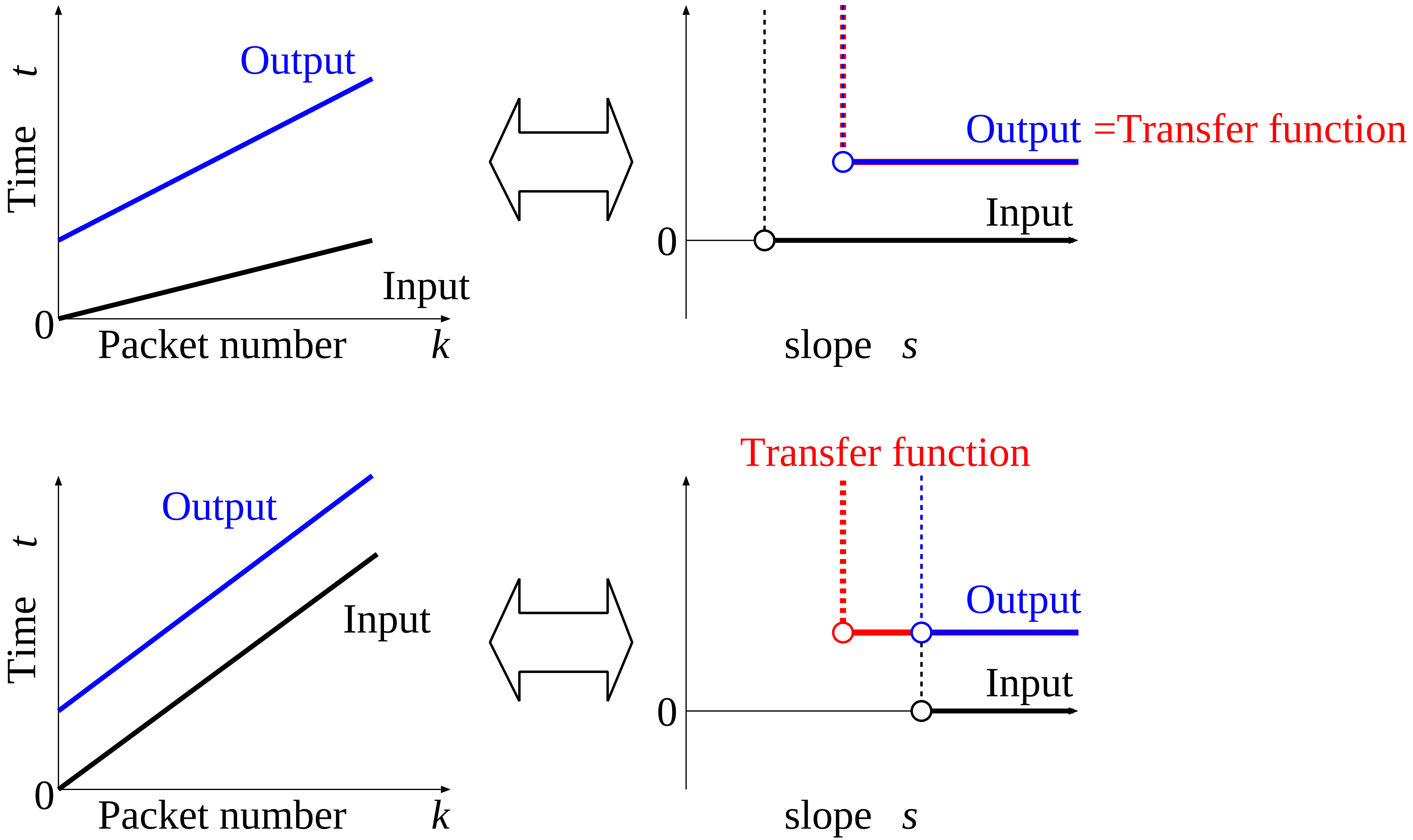
The fast Legendre transform is available.

6. Application — Identification of network system

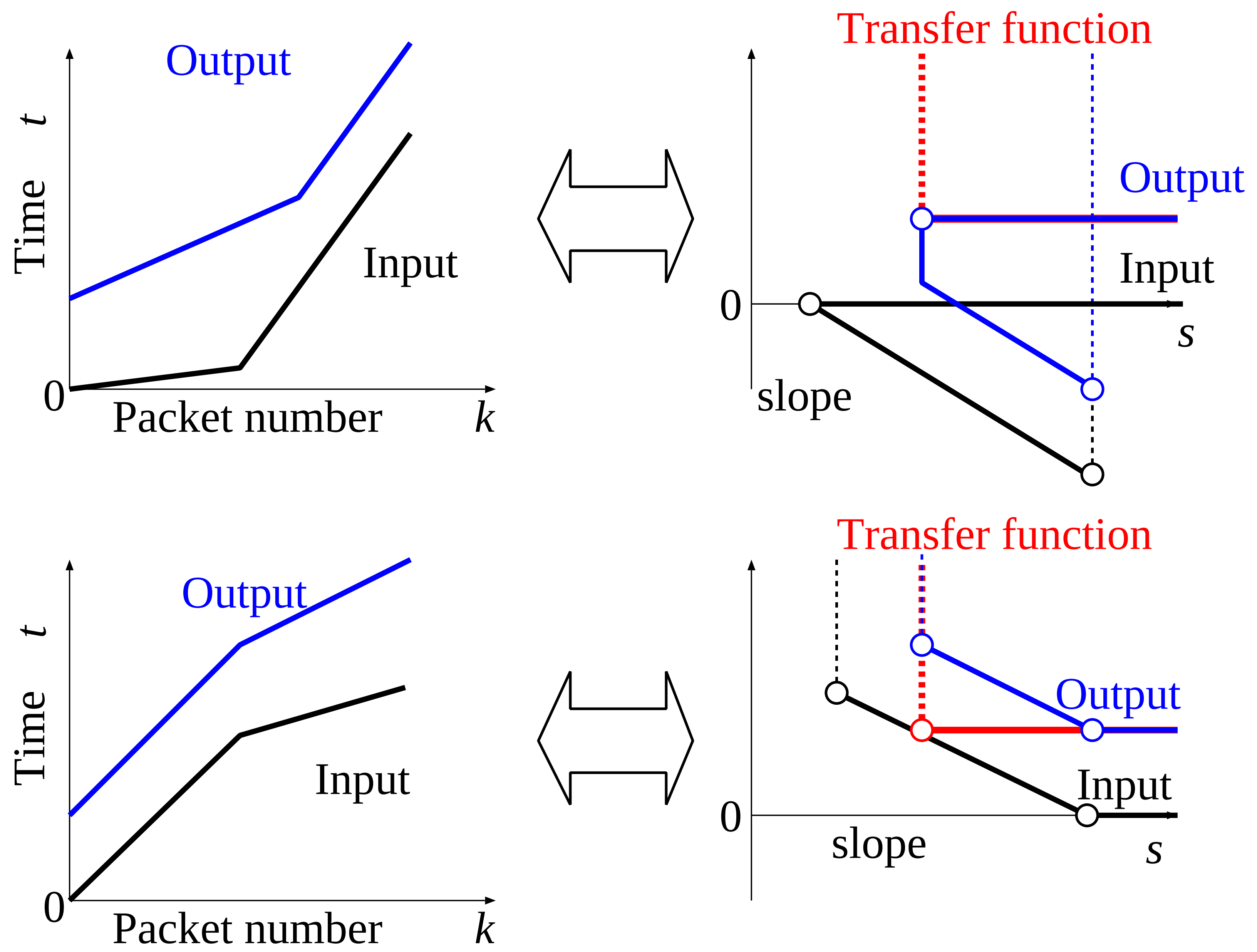
For **traffic analysis**, we need impulse response of **transfer function**. In order to obtain the transfer function, we use **the system identification with deconvolution**.

1. Obtain the input trace $u(k)$ and output trace $y(k)$
2. Calculate **interpolation** $\bar{y}(t)$ and $\bar{u}(t)$.
3. Apply the **extended Legendre transform** to $\bar{u}(t)$ and $\bar{y}(t)$ and obtain $\bar{U}(s)$ and $\bar{Y}(s)$.
4. Calculate the transfer function $\bar{H}(s) = \bar{Y}(s) - \bar{U}(s)$. This is the **deconvoloution process**.
5. Obtain impulse response $\bar{h}(t)$ using the **extended inverse Legendre transform**.

6. Application — Transfer function of Bottleneck router



6. Application — Representation of traffic change



7. Conclusion

- TCP is linear in max-plus algebra.
- The output is given by the convolution of input and transfer function.
- The extended Legendre transform in max-plus algebra corresponds to Fourier transform and z-transform in linear system theory.
- The deconvolution is given by the difference of the transformed input and output.
- Thus, we can identify the transfer function of network system using the extended Legendre transform.

8. Future work

- **We have to confirm the effectiveness using simulation.**
- **We can try to apply this method to any protocols and any networks.**

References

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10. Appendix — Why Legendre transform?

Conventional Fourier transform

$$F(s) = \int f(t)e^{-st}\mathrm{d}t \tag{1}$$

e^{-st} **is based on characters of the group:**

$$f(x+y) = f(x)f(y).$$

Max-plus algebra analog is

$$f(x+y) = f(x) + f(y).$$

Max-plus characters of that group are $-s \cdot x = \log e^{-st}$
Fourier transform in max-plus algebra is

$$F(s) = \max(f(t) - s \cdot t) \tag{2}$$

That is Legendre transform!