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Tests of fit for the von Mises distribution

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Summary

Critical points for the asymptotic distribution of Watson's U^2 statistic are given for testing the hypothesis that a sample comes from the two-parameter von Mises distribution on the circle when one or more of the parameters is unknown. Monte Carlo results show that convergence to the asymptotic distribution is quick. Dependence of the points on the shape parameter is discussed and seen not to cause any problems.

Some key words: Empirical distribution function test; Watson's statistic.

1. Introduction

The von Mises distribution is often used to describe unimodal data on the circumference of a circle. In this paper we give a goodness-of-fit test based on the empirical distribution function statistic U^2 for the hypothesis that a sample comes from this distribution with unknown parameters to be estimated from the data.

Suppose the circle has centre O and radius one, and let a radius OP be measured in radians by the polar coordinate θ from some fixed reference radius ON. Let θ_0 be the coordinate of a radius OA, and let κ be a positive constant. The von Mises density is

$$f(\theta;\, \theta_0, \kappa) = \{2\pi I_0(\kappa)\}^{-1} \exp\{\kappa \cos(\theta - \theta_0)\} \quad (0 < \theta < 2\pi). \tag{1}$$

Here $I_0(\kappa)$ is the imaginary Bessel function of order zero. The distribution has a mode along OA, that is, at $\theta = \theta_0$, and is symmetric about OA; for $\kappa = 0$ the distribution reduces to the uniform distribution around the circle. Suppose a random sample of points $P_1, \ldots, P_n$ on the circumference of the circle has coordinates $\theta_1, \ldots, \theta_n$. We discuss the test of the hypothesis H_0 that the random sample of θ-values comes from the von Mises distribution (1). Four cases may be distinguished according to which parameters, if any, are known:

Case 0: θ_0 and κ are known, so that $f(\theta;\, \theta_0, \kappa)$ is completely specified.
Case 1: θ_0 is unknown and κ is known.
Case 2: θ_0 is known and κ is unknown.
Case 3: Both θ_0 and κ are unknown.

Maximum likelihood estimates of θ_0 and κ are found as follows. Let R be the length of the vector sum or resultant of the vectors OP_i $(i = 1, \ldots, n)$. The estimate $\hat{\theta}$ of θ_0 is the direction of R, the vector sum, and the estimate $\hat{\kappa}$ of κ is the solution to

$$I_1(\kappa)/I_0(\kappa) = R/n, \tag{2}$$

where $I_1(\kappa)$ is the imaginary Bessel function of order one. Tables for solving (2) are given, for example, by Pearson & Hartley (1972, p. 364) and Mardia (1972, p. 297). When OA, that is θ_0, is known, let X be the component of the vector sum along OA; then the estimate of κ now satisfies

$$I_1(\kappa)/I_0(\kappa) = X/n. \tag{3}$$

See § 4 for a discussion of negative estimates of κ in (3).

The test statistic to be discussed is Watson's statistic U^2 defined as follows. Let $F_n(\theta)$ be the empirical distribution function of the n values θ_i, that is, $F_n(\theta)$ is the proportion of $\theta_i \leqslant \theta$, for $0 \leqslant \theta \leqslant 2\pi$.

Define

$$F(\theta;\, \theta_0, \kappa) = \int_0^\theta f(\phi;\, \theta_0, \kappa)\, d\phi.$$

For ease of notation let $F(\theta) \equiv F(\theta;\, \theta_0, \kappa)$. The definition of U^2 is, from Watson (1961) with slight changes in notation,

$$U^2 = n\int_0^{2\pi}\left[F_n(\theta) - F(\theta) - \int_0^{2\pi}\{F_n(\phi) - F(\phi)\}\, dF(\phi)\right]^2 dF(\theta). \tag{4}$$

Clearly U^2 measures the discrepancy between F_n and F. The statistic is an adaptation of the well-known Cramér–von Mises statistic W^2; the adaptation is necessary to make U^2 independent of the choice of origin for θ around the circle.

In § 2, we set out the test procedure. In § 3 we present asymptotic theory and the results of Monte Carlo studies. In § 4, we comment on the dependence of the points on κ. Particular attention is given to large and small κ.

2. THE GOODNESS-OF-FIT TEST

A test of H_0 is made as follows:

(i) for the appropriate case, estimate unknown parameters as described above;
(ii) calculate $z_i = F(\theta_i;\, \theta_0, \kappa)$, where θ_0 and κ are replaced by their estimates if necessary;
(iii) put the z_i in ascending order to obtain $z_{(1)} < \ldots < z_{(n)}$;
(iv) calculate U^2 as

$$U^2 = \Sigma\{z_{(i)} - (2i-1)/(2n)\}^2 - n(\bar{z} - \tfrac{1}{2})^2 + 1/(12n),$$

where $\bar{z} = \Sigma z_i/n$ is the sample average of the z values;
(v) for Case 0 calculate $U^* = (U^2 - 0{\cdot}1/n + 0{\cdot}1/n^2)(1{\cdot}0 + 0{\cdot}8/n)$;
(vi) for Case 0 refer U^* to Table 1; for other cases refer U^2 to the part of Table 1 appropriate to the given case, using the value of κ, or its estimate if necessary, to enter the table; reject H_0 at approximate significance level α if U^* or U^2 exceeds the point given for level α.

For Case 0 exact points for finite n have been calculated by Stephens (1964). These lead to the modification U^* which, together with the asymptotic points, gives essentially an exact test. For other cases the levels for U^2 are approximate since only asymptotic points are given; however, these will be accurate for practical purposes for $n \geqslant 20$. This has been verified by Monte Carlo studies for finite n and is consistent with the behaviour of the null U^2 distribution in many other situations; see, for example, Stephens (1974, 1977). The asymptotic points for U^2 have been obtained from the theoretical calculations of § 3.

Table 1. *Upper tail percentage points for* U^2

κ	0·500	0·25	0·15	0·10	0·05	0·025	0·01	0·005
				Significance level α				
			(a) Case 0: both parameters known					
All κ	0·069	0·105	0·131	0·152	0·187	0·222	0·268	0·304
			(b) Case 1: shape parameter κ known					
0·0	0·047	0·071	0·089	0·105	0·133	0·163	0·204	0·235
0·50	0·048	0·072	0·091	0·107	0·135	0·165	0·205	0·237
1·00	0·051	0·076	0·095	0·111	0·139	0·169	0·209	0·241
1·50	0·053	0·080	0·099	0·115	0·144	0·173	0·214	0·245
2·00	0·055	0·082	0·102	0·119	0·147	0·177	0·217	0·248
4·00	0·058	0·086	0·107	0·124	0·153	0·183	0·224	0·255
∞	0·059	0·089	0·110	0·127	0·157	0·187	0·228	0·259
			(c) Case 2: θ_0 known					
0·0	0·047	0·071	0·089	0·105	0·133	0·163	0·204	0·235
0·50	0·048	0·072	0·091	0·107	0·135	0·165	0·205	0·237
1·00	0·051	0·076	0·095	0·111	0·139	0·169	0·209	0·241
1·50	0·053	0·080	0·100	0·116	0·144	0·174	0·214	0·245
2·00	0·055	0·082	0·103	0·119	0·148	0·177	0·218	0·249
4·00	0·057	0·085	0·106	0·122	0·151	0·181	0·221	0·253
∞	0·057	0·085	0·105	0·122	0·151	0·180	0·221	0·252
			(d) Case 3: neither parameter known					
0·0	0·030	0·040	0·046	0·052	0·061	0·069	0·081	0·090
0·50	0·031	0·042	0·050	0·056	0·066	0·077	0·090	0·100
1·00	0·035	0·049	0·059	0·066	0·079	0·092	0·110	0·122
1·50	0·039	0·056	0·067	0·077	0·092	0·108	0·128	0·144
2·00	0·043	0·061	0·074	0·084	0·101	0·119	0·142	0·159
4·00	0·047	0·067	0·082	0·093	0·113	0·132	0·158	0·178
∞	0·048	0·069	0·084	0·096	0·117	0·137	0·164	0·184

For $\kappa > 4$ use linear interpolation in $1/\kappa$. For Cases 2 and 3 enter the table at the estimate of κ.

3. Theory of the tests

The statistic U^2 may be written as $\int Y_n^2(z)\,dz$, where the integral is over (0, 1). Here

$$Y_n(z) = n^{\frac{1}{2}}\left[F_n(z) - z - \int_0^1 \{F_n(t) - t\}\,dt\right],$$

where F_n is the empirical distribution function of the z_i. The asymptotic behaviour of Y_n has been studied extensively; we give only the details needed for the present calculations. Durbin (1973) gives the theoretical development for both unknown and known parameters.

As $n \to \infty$, Y_n converges weakly to a Gaussian process Y with mean 0 and covariance function $\rho(s,t)$ which depends on the case. For Case 0,

$$\rho(s,t) \equiv \rho_0(s,t) = \min(s,t) - st - \tfrac{1}{2}s(1-s) - \tfrac{1}{2}t(1-t) + \tfrac{1}{12}.$$

For other cases $\rho(s,t) = \rho_0(s,t) - R(s,t)$, where R is $\phi_1(s)\,\phi_1(t)$ for Case 1, $\phi_2(s)\,\phi_2(t)$ for Case 2, $\phi_1(s)\,\phi_1(t) + \phi_2(s)\,\phi_2(t)$ for Case 3, and ϕ_1, ϕ_2 are defined as follows.

Set $M(\kappa) = I_1(\kappa)/I_0(\kappa)$, $N(\kappa) = \{\kappa M(\kappa)\}^{-\frac{1}{2}}$,

$$P(\kappa) = \{d^2 \log I_0(\kappa)/d\kappa^2\}^{-\frac{1}{2}} = \{1 - M(\kappa)/\kappa - M^2(\kappa)\}^{-\frac{1}{2}}.$$

Then

$$\phi_1(s) = \kappa N(\kappa) \int_0^\theta \sin\phi \, f(\phi;\, 0, \kappa)\, d\phi,$$

$$\phi_2(s) = P(\kappa) \left\{ -M(\kappa)\, s + \int_0^\theta \cos\phi \, f(\phi;\, 0, \kappa)\, d\phi \right\},$$

where θ is related to s by $s = F(\theta;\, 0, \kappa)$. The derivation follows the analysis of Stephens (1976) for the normal and exponential distributions.

The asymptotic distribution of U^2 is the same as that of $S = \Sigma\, \omega_i/\lambda_i$, where ω_i are independent χ_1^2 variables and $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ are the eigenvalues of the Fredholm equation

$$f(s) = \lambda \int_0^1 \rho(s, t)\, f(t)\, dt.$$

Effectively, therefore, the problem is to find the eigenvalues λ_i for the different cases and then calculate the distribution of S.

The eigenvalues λ_i are found (Stephens, 1976) by Fourier expansion as follows. Let $\delta_j = (2\pi j)^2$ $(j = 1, 2, \ldots)$. For Case 0 the eigenvalues are two sets of the δ_j. For Case 1 the eigenvalues are one set of the δ_j and the set of roots γ_j of the equation

$$S(\gamma) = 1 + \gamma\, \Sigma\, (a_j^2 + a_j^{*2})/(1 - \gamma/\delta_j) = 0, \tag{5}$$

where

$$a_j = 2^{\frac{1}{2}} \int_0^1 \sin(2\pi j t)\, \phi_1(t)\, dt = 0,$$

$$a_j^* = 2^{\frac{1}{2}} \int_0^1 \cos(2\pi j t)\, \phi_1(t)\, dt = 2^{3/2}\, N(\kappa) \int_0^\pi \cos\{2\pi j F(\theta;\, 0, \kappa)\}\, f^2(\theta;\, 0, \kappa)\, d\theta.$$

For Case 2 the eigenvalues are one set of the δ_i and the set η_i of roots of

$$T(\eta) = 1 + \eta\, \Sigma\, (b_j^2 + b_j^{*2})/(1 - \eta/\delta_j) = 0, \tag{6}$$

where

$$b_j = 2^{\frac{1}{2}} \int_0^1 \sin(2\pi j t)\, \phi_2(t)\, dt$$

$$= 2^{3/2}\, P(\kappa)\, \delta_j^{-\frac{1}{2}} \int_0^\pi \cos\{2\pi j F(\theta;\, 0, \kappa)\} \cos(\theta)\, f(\theta;\, 0, \kappa)\, d\theta,$$

$$b_j^* = 2^{\frac{1}{2}} \int_0^1 \cos(2\pi j t)\, \phi_2(t)\, dt = 0.$$

For Case 3 the eigenvalues are the two sets γ_j, η_j.

In practice we calculate a_j^* and b_j by numerical integration for $j = 1, \ldots, J$, where we find $J = 50$ is a suitable limit. Equations (5) and (6) are then solved numerically for $\gamma_1, \ldots, \gamma_k$ and $\eta_1, \ldots, \eta_k$. The value $k = 40$ works well. Once the eigenvalues for a particular case have been found the percentage points of S are found by Imhof's method, as adapted for an infinite set of eigenvalues by Durbin & Knott (1972).

Since the computations use many approximations it is important to have checks on their numerical accuracy. The cumulants of S were calculated in several ways; details are available from the authors. Having correct values for the cumulants in hand, we fitted Pearson curves to check the accuracy of Table 1.

4. Influence of the shape parameter

Since U^2 is location invariant, its distribution does not depend on θ_0; it does, however, depend on κ, even asymptotically, except in Case 0 where the test is distribution free. In Cases 2 and 3, where κ is unknown, an estimate $\hat{\kappa}$ must be used to enter Table 1. This introduces an extra approximation beyond the asymptotic one since the Table may be entered at the incorrect line. Asymptotically the effect is negligible since $\hat{\kappa}$ is consistent and the points depend continuously on κ. Even for finite samples simulations show that the points depend only slightly on κ for reasonably large κ. In fact, then, little error will result for such κ unless $\hat{\kappa}$ is seriously in error. For small values of κ the points depend more strongly on κ; however in a practical situation in which it appears that κ is small one would probably first test for randomness of the points on the circle. For this test simply use Case 0 above with $\kappa = 0$, $\theta_0 = 0$, calculate U^* as described above, and refer to Table 1.

The levels of the tests outlined here for Case 3 were checked by Monte Carlo simulation based on 10,000 trials for $\kappa = 0{\cdot}5$, 2, 5 and 10 and $n = 20$ and 40 for tests at a nominal level of 0·05, using an expanded version of Table 1. In all cases the attained significance level was between 0·045 and 0·050.

For $n \geqslant 20$ the tests were found to be useful in practice with the attained significance levels being slightly lower than the nominal level for small κ. A similar pattern prevails for other cases.

In Table 1 points are provided for $\kappa = 0$ and $\kappa = \infty$. These points deserve special mention. In each case as $\kappa \to \infty$ the covariance function ρ converges in L^2 to the covariance function of the corresponding case for the normal distribution; see Stephens (1976). The critical points thus also converge. The point $\kappa = 0$ is actually part of the parameter space but when $\kappa = 0$, θ_0 is unidentifiable. For Case 1 the points given correspond to the covariance function obtained as $\kappa \to 0$. For Cases 2 and 3 the parameter space is identified with the plane via $\tau_1 = \kappa \cos\theta_0$, $\tau_2 = \kappa \sin\theta_0$. In Case 3 the problem is then regular at $\tau = (0,0)$ and the asymptotic theory of Durbin (1973) applies. In Case 2 the problem is regular at $\tau = (0,0)$ only if negative estimates of κ are allowed as solutions of (3). Thus the problem is regular if the parameter space is regarded as the line making an angle θ_0 with ON but irregular at $(0,0)$ if the parameter space is taken as a half line making this angle. In practice then when, in Case 2, a negative estimate $\hat{\kappa}$ solves (3) we recommend replacing θ_0 by $\theta_0 + \pi$ and $\hat{\kappa}$ by $\kappa^* = -\hat{\kappa}$. Now follow the procedure as usual using κ^* to enter Table 1.

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