STAT 350

Assignment 2

1. In this problem you will prove that

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty$$

is a density.

(a) Let $I = \int_{-\infty}^{\infty} \phi(x) dx$. Show that

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x)\phi(y)\, dx\, dy.$$

HINT: What is $\int_{-\infty}^{\infty} \phi(y) dy$ in terms of I.

(b) Now if

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dx\, dy$$

do the double integral J in polar co-ordinates ($x = r\cos\theta$, $y = r\sin\theta$) to show $J = 1$.

(c) Deduce that ϕ is a density.

2. Suppose X_1, X_2, X_3 are independent $N(\mu, \sigma^2)$ random variables, so that $X_i = \mu + \sigma Z_i$ with Z_1, Z_2, Z_3 independent standard normals.

(a) If $X^T = (X_1, X_2, X_3)$ and $Z^T = (Z_1, Z_2, Z_3)$ express X in the form $AZ + b$ for a suitable matrix A and vector b.

(b) Show that X is $MVN_3(\mu_X, \Sigma_X)$ and identify μ_X and Σ_X.

(c) Let $Y_i = X_i - \bar{X}$ for $i = 1, 2, 3$ and $Y_4 = \bar{X}$. Show that $Y \sim MVN_4(\mu_Y, \Sigma_Y)$ and find μ_Y and Σ_Y.

3. Working with partitioned matrices. Suppose that the design matrix X is partitioned as $X = [\mathbf{1}|X_1|X_2]$ where X_i has p_i columns.

(a) Write $X^T X$ as a partitioned (3 rows, 3 columns) matrix.

(b) A matrix

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

is called *block diagonal.* Show that A^{-1} exists if and only if each A_i^{-1} exists and that then A^{-1} is block diagonal.

(c) Suppose that $\mathbf{1}^T X_i = 0$ for $i = 1, 2$ and $X_1^T X_2 = 0$. Show that $X^T X$ is block diagonal and give a formula for $(X^T X)^{-1}$.

(d) Suppose $\beta^T = [\beta_0 | \beta_1^T | \beta_2^T]$ is partitioned to conform with the partitioning of X (that is β_0 is a scalar and β_i is a column vector of length p_i for $i = 1, 2$. Let $\tilde\beta_0$ be obtained by fitting

$$Y = \mathbf{1}\beta_0 + \epsilon$$

by least squares, $\tilde\beta_1$ be obtained by fitting

$$Y = X_1\beta_1 + \epsilon$$

and similarly for $\tilde\beta_2$. Let $\hat\beta$ be the usual least squares estimate for

$$Y = X\beta + \epsilon\,.$$

Show that $\hat\beta^T = [\tilde\beta_0 | \tilde\beta_1^T | \tilde\beta_2^T]$.

(e) Let $\hat\mu_i$ be the vectors of fitted values corresponding to the estimates $\tilde\beta_i$ for $i = 1, 2, 3$. Show that for $i \neq j$ we have $\hat\mu_i \perp \hat\mu_j$.

(f) For the design matrix X_b of the first assignment identify X_1 and X_2 and verify the orthogonality condition of this problem.

4. Page 321. Problem 7.33 parts a, b, e and f, 7.34 and 7.35 part a.

DUE: Wednesday, 3 February