STAT 350: Lecture 16

Reading: There is no really relevant part of the text.

Theory of F and t tests

Independence: If $U_1, U_2, \ldots U_k$ are random variables then we call $U_1, \ldots, U_k$ independent if

$$P(U_1 \in A_1, \ldots, U_k \in A_k) = P(U_1 \in A_1) \times \cdots \times P(U_k \in A_k)$$

for any sets $A_1, \ldots, A_k$.

We usually either:

- Assume independence because there is no physical way for the value of any of the random variables to influence any of the others.

  OR

- We prove independence.

How do we prove independence? We use the notion of a **joint density**.

We say $U_1, \ldots, U_k$ have joint density function $f = f(u_1, \ldots, u_k)$ if

$$P((U_1, \ldots, U_k) \in A) = \int \cdots \int_A f(u_1, \ldots, u_k) du_1 \cdots du_k$$

We are interested here in joint densities because independence of $U_1, \ldots, U_k$ is equivalent to

$$f(u_1, \ldots, u_k) = f_1(u_1) \times \cdots \times f_k(u_k)$$

for some densities $f_1, \ldots, f_k$. In this case f_i is the density of U_i.

[ASIDE: notice that for an independent sample the joint density is the likelihood function!]

Application to Normals: Standard Case

If

$$Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \sim MVN(0, I_{n\times n})$$

then the joint density of Z, denoted $f_Z(z_1, \ldots, z_n)$ is

$$f_Z(z_1, \ldots, z_n) = \phi(z_1) \times \cdots \times \phi(z_n)$$

where

$$\phi(z_i) = \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2}$$

So

$$
\begin{aligned}
f_Z &= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} z_i^2\right\} \\
&= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2} z^T z\right\}
\end{aligned}
$$

where

$$
z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}
$$

Application to Normals: General Case

If $X = AZ + \mu$ **and** A is invertible then for any set $B \in R^n$ we have

$$
\begin{aligned}
P(X \in B) &= P(AZ + \mu \in B) \\
&= P(Z \in A^{-1}(B - \mu)) \\
&= \int \cdots \int_{A^{-1}(B-\mu)} (2\pi)^{-n/2} \exp\left\{-\frac{1}{2} z^T z\right\} dz_1 \cdots dz_n
\end{aligned}
$$

Make the change of variables $x = Az + \mu$ in this integral to get

$$
P(X \in B) = \int \cdots \int_B (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\left(A^{-1}(x-\mu)\right)^T \left(A^{-1}(x-\mu)\right)\right\} J(x) dx_1 \cdots dx_n
$$

where $J(x)$ denotes the Jacobian of the transformation

$$
J(x) = J(x_1, \ldots, x_n) = \left|\det\left(\frac{\partial z_i}{\partial x_j}\right)\right| = \left|\det\left(A^{-1}\right)\right|
$$

Algebraic manipulation of the integral then gives

$$
P(X \in B) = \int \cdots \int_B (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\} |\det A^{-1}| dx_1 \cdots dx_n
$$

where

$$
\begin{aligned}
\Sigma &= AA^T \\
\Sigma^{-1} &= \left(A^{-1}\right)^T \left(A^{-1}\right) \\
\det \Sigma^{-1} &= \left(\det A^{-1}\right)^2 \\
&= \frac{1}{\det \Sigma}
\end{aligned}
$$

The conclusion of this algebra is that the $MVN(\mu, \Sigma)$ density is

$$(2\pi)^{-n/2} \exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}(\det\Sigma)^{-1/2}$$

What if A is not invertible? Ans: there is no density.
How do we apply this density?
Suppose

$$X = \left[\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right]$$

and

$$\Sigma = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right]$$

If $\Sigma_{12} = 0$ then

1. $\Sigma_{21} = 0$

2. In the homework you will verify that

$$\Sigma^{-1} = \left[\begin{array}{c|c} \Sigma_{11}^{-1} & 0 \\ \hline 0 & \Sigma_{22}^{-1} \end{array}\right]$$

3. Writing

$$x = \left[\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right]$$

and

$$\mu = \left[\begin{array}{c} \mu_1 \\ \hline \mu_2 \end{array}\right]$$

we find

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = (x_1-\mu_1)^T\Sigma_{11}^{-1}(x_1-\mu_1) + (x_2-\mu_2)^T\Sigma_{22}^{-1}(x_2-\mu_2)$$

4. So, if $n_1 = \dim(X_1)$ and $n_2 = \dim(X_2)$ we see that

$$\begin{aligned} f_X(x_1, x_2) = (2\pi)^{-n_1/2} &\exp\left\{-\frac{1}{2}(x_1-\mu_1)^T\Sigma_{11}^{-1}(x_1-\mu_1)\right\} \\ &\times (2\pi)^{-n_2/2}\exp\left\{-\frac{1}{2}(x_2-\mu_2)^T\Sigma_{22}^{-1}(x_2-\mu_2)\right\} \end{aligned}$$

so that X_1 and X_2 are independent.

Summary: If $\text{Cov}(X_1, X_2) = \text{E}[(X_1-\mu_1)(X_2-\mu_2)^T] = 0$ then X_1 is independent of X_2.
Warning: This only works provided

$$X = \left[\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right] \sim MVN(\mu, \Sigma)$$

Fact: However, it works even if Σ is singular, but you can't prove it as easily using densities.
Application:

$$\begin{aligned}
\hat{\mu} &= X\hat{\beta} = X(X^TX)^{-1}X^TY \\
&= X\beta + H\epsilon \\
\hat{\epsilon} &= Y - X\hat{\beta} \\
&= \epsilon - H\epsilon \\
&= (I-H)\epsilon
\end{aligned}$$

So

$$\left[\frac{\hat{\mu}}{\hat{\epsilon}}\right] = \underbrace{\sigma\left[\frac{H}{I-H}\right]}_{A}\frac{\epsilon}{\sigma} + \underbrace{\left[\frac{\mu}{0}\right]}_{b}$$

Hence

$$\left[\frac{\hat{\mu}}{\hat{\epsilon}}\right] \sim MVN\left(\left[\frac{\mu}{0}\right]; AA^T\right)$$

Now

$$A = \sigma\left[\frac{H}{I-H}\right]$$

so

$$\begin{aligned}
AA^T &= \sigma^2\left[\frac{H}{I-H}\right]\begin{bmatrix} H^T & (I-H)^T \end{bmatrix} \\
&= \sigma^2\begin{bmatrix} HH & H(I-H) \\ (I-H)H & (I-H)(I-H) \end{bmatrix} \\
&= \sigma^2\begin{bmatrix} H & H-H \\ H-H & I-H-H+HH \end{bmatrix} \\
&= \sigma^2\begin{bmatrix} H & 0 \\ 0 & I-H \end{bmatrix}
\end{aligned}$$

The 0s **prove** that $\hat{\epsilon}$ and $\hat{\mu}$ are independent. It follows that $\hat{\mu}^T\hat{\mu}$, the regression sum of squares (not adjusted) is independent of $\hat{\epsilon}^T\hat{\epsilon}$, the Error sum of squares.

Joint Densities

Suppose Z_1 and Z_2 are independent standard normals. In class I said that their joint density was

$$f(z_1, z_2) = \frac{1}{2\pi}\exp(-(z_1^2 + z_2^2)/2).$$

Here I want to show you the meaning of joint density by computing the density of a χ^2_2 random variable.

Let $U = Z_1^2 + Z_2^2$. By definition U has a χ^2 distribution with 2 degrees of freedom. The cumulative distribution function of U is

$$F(u) = P(U \le u).$$

For $u \le 0$ this is 0 so take $u \ge 0$. The event that $U \le u$ is the same as the event that the point (Z_1, Z_2) is in the circle centered at the origin and having radius $u^{1/2}$, that is, if A is the circle of this radius then

$$F(u) = P((Z_1, Z_2) \in A).$$

By definition of density this is a double integral

$$\int\int_A f(z_1, z_2)\, dz_1\, dz_2 .$$

You do this integral in polar co-ordinates. Letting $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$ we see that

$$f(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\exp(-r^2/2).$$

The Jacobian of the transformation is r so that $dz_1\, dz_2$ becomes $r\, dr\, d\theta$. Finally the region of integration is simply $0 \le \theta \le 2\pi$ and $0 \le r \le u^{1/2}$ so that

$$\begin{aligned} P(U \le u) &= \int_0^{u^{1/2}} \int_0^{2\pi} \frac{1}{2\pi}\exp(-r^2/2) r\, dr\, d\theta \\ &= \int_0^{u^{1/2}} r\exp(-r^2/2) dr \\ &= -\exp(-r^2/2)\Big|_0^{u^{1/2}} \\ &= 1 - \exp(-u/2). \end{aligned}$$

The density of U can be found by differentiating to get

$$f(u) = \frac{1}{2}\exp(-u/2)$$

which is the exponential density with mean 2. This means that the χ_2^2 density is really an exponential density.