STAT 350

Assignment 2: Solutions

1. In this problem you will prove that

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty$$

is a density.

(a) Let $I = \int_{-\infty}^{\infty} \phi(x)dx$. Show that

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x)\phi(y)\, dx\, dy.$$

HINT: What is $\int_{-\infty}^{\infty} \phi(y)dy$ in terms of I.

Solution

We have

$$\begin{aligned} I^2 &= I \int \phi(y)dy \qquad \text{by change of variables} \\ &= \int I\phi(y)dy \\ &= \int\int \phi(x)dx\phi(y)dy \\ &= \int\int \phi(x)\phi(y)dxdy \\ &= J \end{aligned}$$

(b) Now if

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dx\, dy$$

do the double integral J in polar co-ordinates ($x = r\cos\theta$, $y = r\sin\theta$) to show $J = 1$.

Solution

When you do an integral in polar co-ordinates you have to: replace each x in the integrand with $r\cos\theta$ and each y with $r\sin\theta$, replace $dxdy$ with $|Jacobian|drd\theta$, and find the set of r, θ values which correspond to the set of x, y values over which we are integrating. The Jacobian is the absolute value of the determinant filled up with derivatives of (x, y) with respect to r and θ. This 2 by 2 matrix has determinant r. The value of r, being a distance from the origin is in the range 0 to ∞ while the angle in the plane is measured over any interval of length 2π such as $[0, 2\pi)$. This makes

$$J = \int_0^{\infty} \int_0^{2\pi} \frac{1}{2\pi} \exp(-r^2/2) r dr d\theta\,.$$

The θ integral gives2π leaving

$$J = \int_0^{\infty} r\exp(-r^2/2)\, dr = 1\,.$$

(c) Deduce that ϕ is a density.

Solution

All you have to do is prove that $\phi \geq 0$ and $\int \phi = 1$. But ϕ is clearly positive. Thus $I > 0$ and since $I^2 = J = 1$ we have $I = 1$.

2. Suppose X_1, X_2, X_3 are independent $N(\mu, \sigma^2)$ random variables, so that $X_i = \mu + \sigma Z_i$ with Z_1, Z_2, Z_3 independent standard normals.

(a) If $X^T = (X_1, X_2, X_3)$ and $Z^T = (Z_1, Z_2, Z_3)$ express X in the form $AZ + b$ for a suitable matrix A and vector b.

Solution

We have $A = \sigma I$ and $b^T = [\mu, \mu, \mu]$.

(b) Show that X is $MVN_3(\mu_X, \Sigma_X)$ and identify μ_X and Σ_X.

Solution

The definition of MVN is that X be of the form $AZ + b$ and then $\mu_X = b$ and $\Sigma_X = AA^T$. So $\mu_x^T = [\mu, \mu, \mu]$ and $\Sigma_X = \sigma^2 I$.

(c) Let $Y_i = X_i - \bar{X}$ for $i = 1, 2, 3$ and $Y_4 = \bar{X}$. Show that $Y \sim MVN_4(\mu_Y, \Sigma_Y)$ and find μ_Y and Σ_Y.

Solution

Let B be the matrix

$$B = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Then $Y = BX$ so Y is $MVN_r(B\mu_X, B\Sigma_X B^T)$.

Arithmetically we find

$$\mu_Y^T = (B\mu_x)^T = [0, 0, 0, \mu]$$

and

$$\Sigma_Y = \sigma^2 \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 \\ -1/3 & 2/3 & -1/3 & 0 \\ -1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}.$$

3. Working with partitioned matrices. Suppose that the design matrix X is partitioned as $X = [\mathbf{1}|X_1|X_2]$ where X_i has p_i columns.

(a) Write $X^T X$ as a partitioned (3 rows, 3 columns) matrix.

Solution

$$X^T X = \begin{bmatrix} n & 1^T X_1 & 1^t X_2 \\ X_1^T 1 & X_1^T X_1 & X_1^T X_2 \\ X_2^T 1 & X_2^T X_1 & X_2^T X_2 \end{bmatrix}.$$

(b) A matrix

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

is called *block diagonal.* Show that A^{-1} exists if and only if each A_i^{-1} exists and that then A^{-1} is block diagonal.

Solution

Check by multiplying that

$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} (A_1)^{-1} & 0 & 0 \\ 0 & (A_2)^{-1} & 0 \\ 0 & 0 & (A_3)^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

This shows that if each A_i is invertible then so is A. I notice in marking that many of you assumed that each A_i was a scalar so that $\frac{1}{A_i}$ would make sense. That would make A diagonal, not block diagonal and would be useless for the next part of this question. To do the converse suppose that B is A^{-1} and partition B into a 3×3 array with entries B_{ij}. Multiply AB and set this equal to the identity. You get 9 equations like $A_1 B_{1,1} = I$ and $A_1 B_{1,2} = 0$. The first such equation shows that A_1 must be invertible and that $B_{1,1}$ must be the inverse of A_1. The second equation then shows (because we now know that A_1 is invertible that $B_{1,2} = 0$. Continue like this.

(c) Suppose that $\mathbf{1}^T X_i = 0$ for $i = 1, 2$ and $X_1^T X_2 = 0$. Show that $X^T X$ is block diagonal and give a formula for $(X^T X)^{-1}$.

Solutions

The conditions show that all the off-diagonal blocks are 0, remembering that $(AB)^T = B^T A^T$. Thus

$$(X^T X)^{-1} = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & (X_1^T X_1)^{-1} & 0 \\ 0 & 0 & (X_2^T X_2)^{-1} \end{bmatrix}$$

Many students appeared not to realize that $\mathbf{1}^T \mathbf{1} = n$ because $\mathbf{1}$ is a **vector** all of whose entries equal the number 1.

(d) Suppose $\beta^T = [\beta_0 | \beta_1^T | \beta_2^T]$ is partitioned to conform with the partitioning of X (that is β_0 is a scalar and β_i is a column vector of length p_i for $i = 1, 2$. Let $\tilde{\beta}_0$ be obtained by fitting

$$Y = \mathbf{1}\beta_0 + \epsilon$$

by least square, $\tilde{\beta}_1$ be obtained by fitting

$$Y = X_1 \beta_1 + \epsilon$$

and similarly for $\tilde{\beta}_2$. Let $\hat{\beta}$ be the usual least squares estimate for

$$Y = X\beta + \epsilon\,.$$

Show that $\hat{\beta}^T = [\tilde{\beta}_0|\tilde{\beta}_1^T|\tilde{\beta}_2^T]$.

Solutions

Multiply out the partitioned matrix $\hat{\beta} = (X^TX)^{-1}X^TY$ to get

$$\hat{\beta} = \begin{bmatrix} \bar{Y} \\ (X_1^TX_1)^{-1}X_1^TY \\ (X_2^TX_2)^{-1}X_2^TY \end{bmatrix} = \begin{bmatrix} \tilde{\beta}_0 \\ \tilde{\beta}_1 \\ \tilde{\beta}_2 \end{bmatrix}.$$

(e) Let $\hat{\mu}_i$ be the vectors of fitted values corresponding to the estimates $\tilde{\beta}_i$ for $i = 1, 2, 3$. [Note this should be 0,1,2] Show that for $i \neq j$ we have $\hat{\mu}_i \perp \hat{\mu}_j$.

Solution

$$\hat{\mu}_1^T\hat{\mu}_2 = \tilde{\beta}_1^TX_1^TX_2\tilde{\beta}_2 = 0$$

because the centre term $X_1^TX_2 = 0$. You must check similar formulas for $\hat{\mu}_0^T\hat{\mu}_1$ and so on.

(f) For the design matrix X_b of the first assignment identify X_1 and X_2 and verify the orthogonality condition of this problem.

Solution

In the solution set for assignment the first column of X_b is 1, the second is X_1 and the other two columns are X_2. Now just multiply things like $X_1^TX_2$ to make sure you get 0.

4. Page 321. Problem 7.33 parts a, b, e and f, 7.34 and 7.35 part a.

Solution

I used the following SAS code to find the mean age and to compute the correlation between age and age cubed:

```
data product;
 infile "CH07PR33.DAT";
 input product age;
proc means ;
proc glm;
  model age = age*age*age;
run;
```

Part of the output is

```
Variable   N          Mean        Std Dev        Minimum        Maximum
-----------------------------------------------------------------------
PRODUCT   18   104.2222222      5.4292839     93.0000000    112.0000000
AGE       18    40.0000000     13.2842233     20.0000000     60.0000000
-----------------------------------------------------------------------
```

```
Dependent Variable: AGE
                                  Sum of           Mean
Source               DF          Squares         Square  F Value     Pr > F
Model                 1       2799.13108     2799.13108   222.96     0.0001
Error                16        200.86892       12.55431
Corrected Total      17       3000.00000
              R-Square             C.V.        Root MSE            AGE Mean
              0.933044         8.858015         3.54321             40.0000

                                  T for H0:     Pr > |T|    Std Error of
Parameter          Estimate    Parameter=0                   Estimate

INTERCEPT      24.61714029          18.56       0.0001      1.32618815
AGE*AGE*AGE     0.00018313          14.93       0.0001      0.00001226
```

Since $r^2 = 0.933$ and the slope is positive we find that the correlation between age and age cubed is 0.966.

Having found the mean age I did the rest of the problem with:

```
data product;
 infile "CH07PR33.DAT";
 input product age;
 agec = age-40;
 agec2 = agec*agec;
 agec3 = agec*agec*agec;
proc print;
proc glm;
  model product= agec agec2 agec3;
output out=prodout r=resid p=fitted ;
run;
proc glm;
  model agec3 = agec;
run;
proc rank normal=vw data=prodout out=prodout2;
  var resid;
  ranks normscr;
run;
proc gplot data=prodout2;
 plot resid*agec;
 plot resid*fitted;
 plot resid*normscr;
run;
```

obtaining the (edited) output:

OBS	PRODUCT	AGE	AGEC	AGEC2	AGEC3
1	97	20	-20	400	-8000
2	93	20	-20	400	-8000
3	99	25	-15	225	-3375
4	105	25	-15	225	-3375
5	109	30	-10	100	-1000
6	106	30	-10	100	-1000
7	109	35	-5	25	-125
8	111	35	-5	25	-125
9	100	40	0	0	0
10	105	40	0	0	0
11	97	45	5	25	125
12	101	45	5	25	125
13	105	50	10	100	1000
14	103	50	10	100	1000
15	105	55	15	225	3375
16	109	55	15	225	3375
17	112	60	20	400	8000
18	110	60	20	400	8000

Dependent Variable: PRODUCT

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	351.680014	117.226671	10.98	0.0006
Error	14	149.431097	10.673650		
Corrected Total	17	501.111111			

R-Square	C.V.	Root MSE	PRODUCT Mean
0.701800	3.134701	3.26706	104.222

Source	DF	Type I SS	Mean Square	F Value	Pr > F
AGEC	1	124.033333	124.033333	11.62	0.0042
AGEC2	1	2.955267	2.955267	0.28	0.6070
AGEC3	1	224.691414	224.691414	21.05	0.0004

Source	DF	Type III SS	Mean Square	F Value	Pr > F
AGEC	1	92.201230	92.201230	8.64	0.0108
AGEC2	1	2.955267	2.955267	0.28	0.6070
AGEC3	1	224.691414	224.691414	21.05	0.0004

Parameter	Estimate	T for H0: Parameter=0	Pr > |T|	Std Error of Estimate
INTERCEPT	104.6839827	89.66	0.0001	1.16751237

AGEC	-0.4591751	-2.94	0.0108	0.15623072
AGEC2	-0.0027706	-0.53	0.6070	0.00526534
AGEC3	0.0022458	4.59	0.0004	0.00048948

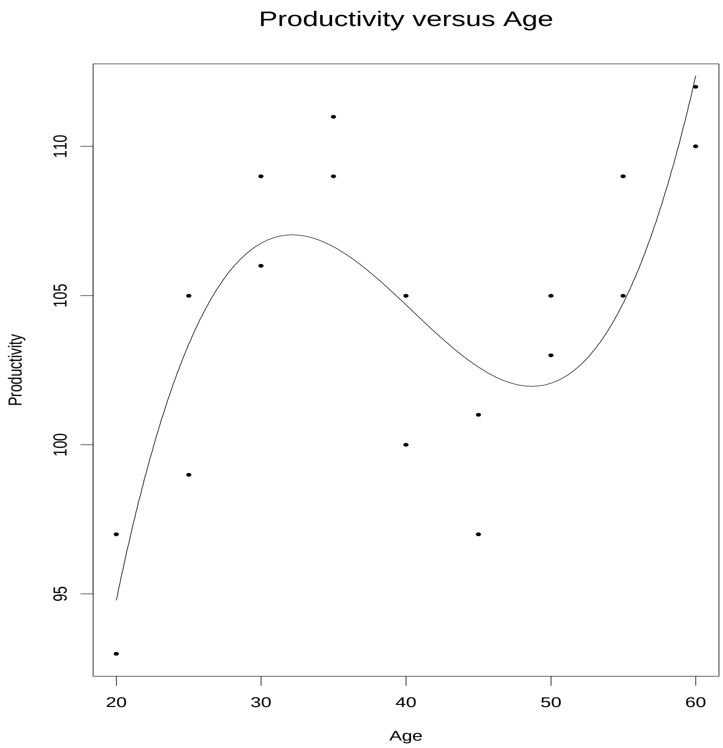
Dependent Variable: AGEC3

	Sum of	Mean	
R-Square	C.V.	Root MSE	AGEC3 Mean
0.854233	9999.99	1668.64	0

Parameter	Estimate	T for H0: Parameter=0	Pr > |T|	Std Error of Estimate
INTERCEPT	0.0000000	0.00	1.0000	393.3033181
AGEC	295.0000000	9.68	0.0001	30.4651440

We are now in a position to answer the individual questions.

7.33 a) The coefficients are in the print-out. Multiple $R^2 = 0.702$. Here is a plot, in SPlus, of the data and fitted values. It is adequate, more or less, although there is a sense that the cubic is too low around 35 years of age and too high around 45. There is a danger of overinterpretation, however.



b) This question asks you to test $H_o : \beta_1 = \beta_2 = \beta_3 = 0$, against the alternative that at least one of these coefficients is not 0. The overall F test does exactly this:

$F = 10.98$ and $P = 0.006$ Since $P < 0.01$ you would reject at the 1% level. The conclusion is that there is some relation between age and productivity.

e) The fitted regression function is

$$\text{Production} \;= 104.68 - 0.459(\text{Age}-40) - 0.00278(\text{Age}-40)^2 + 0.00225(\text{Age}-40)^3$$

You can expand out the powers if you want but it's not really useful. The correlation between Age and Age cubed is 0.966 while that between centered age and centered age cubed is $\sqrt{0.854} = 0.924$ which makes the centering not very useful for decreasing the correlation between $\hat\beta_3$ and $\hat\beta_1$. However, the centering is more numerically stable and lowers the correlation between x and x^3 quite a bit.

7.34a To carry out this test you need to compare the linear model with this cubic model. This can be done by rerunning SAS with the model statement `model product=agec` but you can also do it with the output above. The Type I SS table shows the extra sums of squares for comparing a sequence of models:

Model	ESS
cubic	149.431097
quadratic	149.431097 + 224.691414 =374.122511
linear	374.122511 + 2.955267

The extra sum of squares due to quadratic and cubic terms is then the sum of the last two lines in the Type I SS table which is 227.646681. There are two more parameters in the cubic model than in the linear model so there are 2 degrees of freedom for the extra SS. The F statistic is

$$\frac{227.646681/2}{10.673650} = 10.66$$

on 2, 14 degrees of freedom. (The denominator is just the MSE for the cubic model.) You would need to use F tables to decide whether or not to reject. In fact P is about 0.0015 so you cannot discard the quadratic and cubic terms.

b) This can be done using the last line of either the Type 1 or Type 3 SS tables since these lines both compare the cubic and quadratic models. The F statistic is 21.05 with a P value of 0.0004 so you definitely cannot drop the cubic term. Null is $\beta_3 = 0$, alternative is $\beta_3 \neq 0$ and the conclusion is that $\beta_3 \neq 0$.

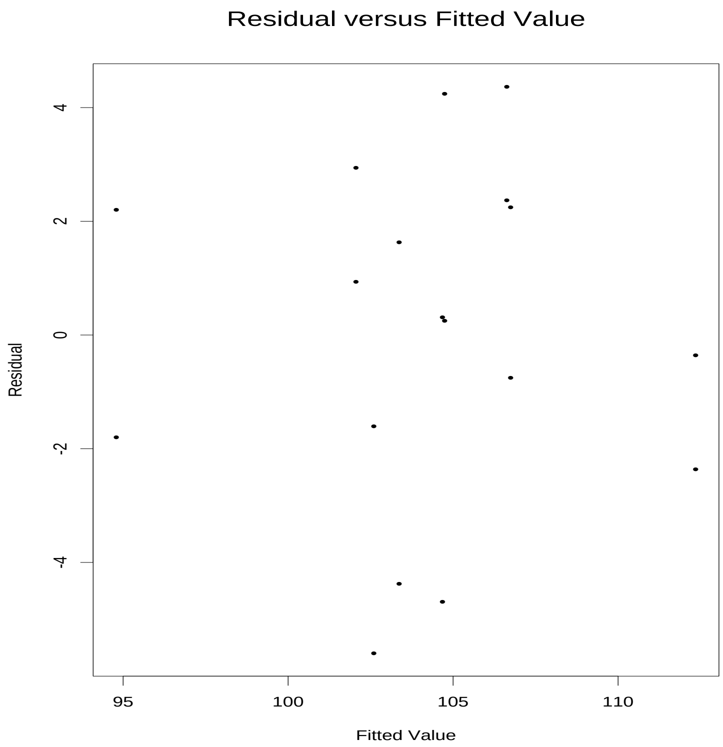
7.35 a) The plots are prepared by the SAS code above. Here are the SPlus versions. They were prepared with the following SPlus code:

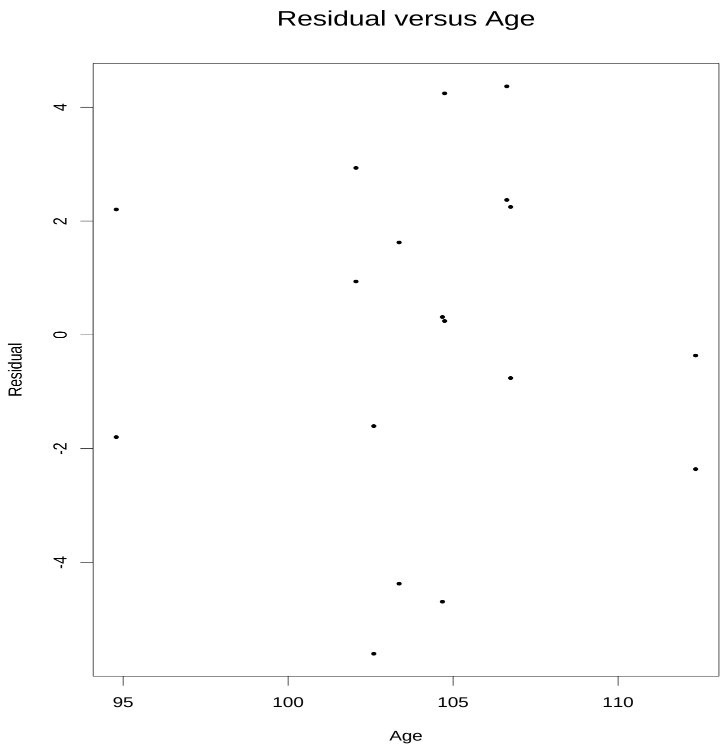
```
#
# Read in the data into a matrix (technically
# a data frame)
#
data <- read.table("CH07PR33.DAT")
#
```

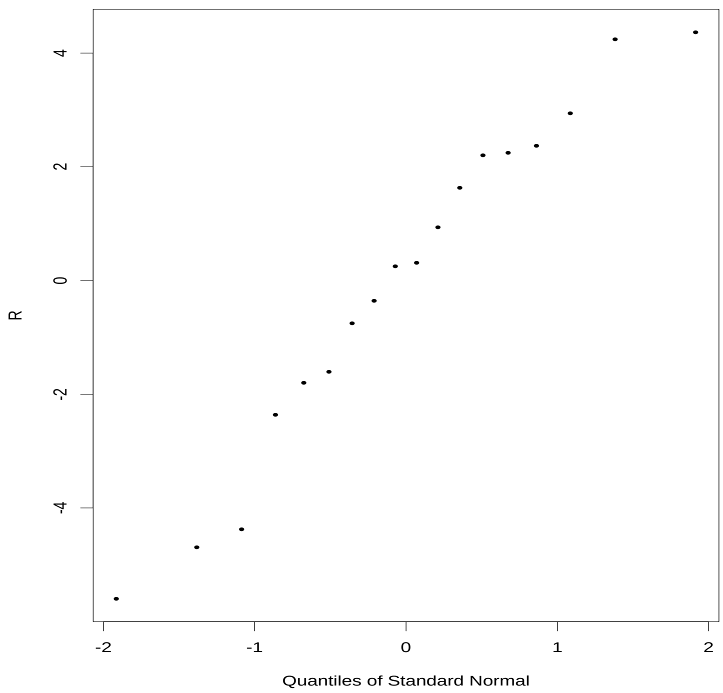
```
# Add a new column to the matrix which has the Ages minus
# the mean age
#
data <- cbind(data,data[,2]-mean(data[,2]))
#
# Give names to the variables in the data frame
#
names(data) <- c("Product","Age","AgeCen")
#
# Compute the mean age for later use
#
m <- mean(data$Age)
#
# make the variables in the data frame available for general
# use without having to say things like data$Age
# which names the Age component of the data frame called data
#
attach(data)
#
# Regress the variable Product on the first three
# powers of the centered Age. Keep the results of
# the regression in a variable called fit
#
fit <- lm( Product ~ AgeCen+AgeCen^2 + AgeCen^3,data=data)
#
# store the residuals in R
#
R <- resid(fit)
#
# store the fitted values in Fit
Fit <- predict(fit)
#
#  Make a postscript plot in a file fit.ps. The horizontal=F
# causes the image to be in portrait orientation, that is,
# in the usual direction (not turned sideways) on a piece
# of paper.
#
postscript("fit.ps",horizontal=F)
#
#  Plot Productivity versus Age.
#
plot(Age, Product,xlab="Age", ylab="Productivity",
   main="Productivity versus Age")
#
```

```
# Add a plot of the fitted cubic
#
# First make a vector full of ages
#
xp <- seq(from=min(Age),to=max(Age),length=500)
#
# now compute the corresponding fitted values
# The regression coefficients are obtained using
# the function coef.
#
co <- coef(fit)
yp <- co[1] + co[2]*(xp-m)
        + co[3]*(xp-m)^2 + co[4]*(xp-m)^3
#
# Add a curve to the plot
#
lines(xp,yp)
#
# finish this plot
#
dev.off()

#
# Now plot residuals against Fitted value
#
postscript("r_v_fit.ps",horizontal=F)
plot(Fit,R,xlab="Fitted Value",
        ylab="Residual",main="Residual versus Fitted Value")
dev.off()
#
# Now plot residuals against Age
#
postscript("r_v_age.ps",horizontal=F)
plot(Fit,R,xlab="Age",ylab="Residual",main="Residual versus Age")
dev.off()
#
# Make a QQ plot
#
postscript("qq.ps",horizontal=F)
qqnorm(R)
dev.off()
```


Residual versus Fitted Value
Residual
4
2
0
-2
-4
95
100
105
110
Fitted Value


Residual versus Age
Residual
4
2
0
-2
-4
95
100
105
110
Age


R
4
2
0
-2
-4
-2
-1
0
1
2
Quantiles of Standard Normal