STAT 350

Assignment 5

Part A

1. In this question you will derive some of the formulas for case deleted statistics. Suppose that X is an $n \times p$ design matrix. Let x_i^T be the i^{th} row of X (so that, x_i is a column vector or dimension p). Let $X_{(i)}$ be the design matrix with case i deleted and $Y_{(i)}$ the vector of $n-1$ responses with case i deleted.

   (a) Use partitioned matrices to show
   $$X_{(i)}^T Y_{(i)} = X^T Y - x_i Y_i \, .$$
   Write
   $$X = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix}$$
   and
   $$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$
   Then
   $$X^T Y = [x_1 | x_2 | \cdots | x_n] \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \sum_{j=1}^n x_j Y_j$$
   Since $X_{(i)}$ and $Y_{(i)}$ simply omit the ith rows we see
   $$X_{(i)}^T Y_{(i)} = X^T Y - x_i Y_i \, .$$

   (b) Show
   $$X_{(i)}^T X_{(i)} = X^T X - x_i x_i^T \, .$$
   We have
   $$X^T X = [x_1 | x_2 | \cdots | x_n] \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix} = \sum_{j=1}^n x_j x_j^T$$
   Again omitting the ith term we find
   $$X_{(i)}^T X_{(i)} = X^T X - x_i x_i^T \, .$$

(c) Suppose B is an invertible symmetric $p \times p$ matrix and v is a column vector of dimension p. Show, by direct multiplication, that $(B - vv^T)^{-1}$ is of the form

$$B^{-1} + rB^{-1}vv^TB^{-1}$$

and give a formula for the scalar r.

Simply multiply

$$(B - vv^T)(B^{-1} + rB^{-1}vv^TB^{-1}) = I + rvv^TB^{-1} - vv^TB^{-1} - rvv^TB^{-1}vv^TB^{-1}$$

The trick most students missed is that in the last term the quantity $v^TB^{-1}v$ is a scalar, that is, just a single number (check that it is a 1×1 matrix). It may help to give it a name like s to see that the last term is

$$rsvv^TB^{-1} .$$

Thus

$$(B - vv^T)(B^{-1} + rB^{-1}vv^TB^{-1}) = I + (r - 1 - rs)vv^TB^{-1}$$

which is the identity matrix if

$$r - rs - 1 = 0 .$$

Solving gives $r = 1/(1 - s) = 1/(1 - v^TB^{-1}v)$. Many students got a formula like

$$r = (1 - vv^TB^{-1})^{-1}$$

which is a $p \times p$ matrix not a scalar.

(d) Apply to previous part to show

$$(X_{(i)}^T X_{(i)})^{-1} = (X^TX)^{-1} + r_i(X^TX)^{-1}x_ix_i^t(X^TX)^{-1}$$

and give a formula for r_i in terms of the leverage $h_{ii} = x_i^T(X^TX)^{-1}x_i$.

Put $B = X^TX$ and $v = x_i$. You get

$$r_i = 1/(1 - x_i^T(X^TX)^{-1}x_i) = 1/(1 - h_{ii})$$

and

$$(X_{(i)}^T X_{(i)})^{-1} = (X^TX)^{-1} + r_i(X^TX)^{-1}x_ix_i^T(X^TX)^{-1} .$$

(e) Show $1 + h_{ii}r_i = r_i$.

We have

$$(1 + r_ih_{ii}) = \frac{1 - h_{ii}}{1 - h_{ii}} + \frac{h_{ii}}{1 - h_{ii}} = \frac{1}{1 - h_{ii}} = r_i$$

(f) Show that

$$\begin{aligned}\hat\beta_{(i)} &= \hat\beta + r_i(X^TX)^{-1}x_ix_i^T\hat\beta - (1+h_{ii}r_i)(X^TX)^{-1}x_iY_i \\ &= \hat\beta - r_i(X^TX)^{-1}x_i\hat\epsilon_i\,.\end{aligned}$$

$$\begin{aligned}\hat\beta_{(i)} &= (X_{(i)}^TX_{(i)})^{-1}X_{(i)}^TY_{(i)} \\ &= \left[(X^TX)^{-1} + r_i(X^TX)^{-1}x_ix_i^T(X^TX)^{-1}\right]\left[X^TY - x_iY_i\right] \\ &= \hat\beta + r_i(X^TX)^{-1}x_ix_i^T(X^TX)^{-1}X^TY \\ &\quad -(X^TX)^{-1}x_iY_i - r_i(X^TX)^{-1}x_ix_i^T(X^TX)^{-1}x_iY_i \\ &= \hat\beta + r_i(X^TX)^{-1}x_ix_i^T\hat\beta \\ &\quad -(X^TX)^{-1}x_iY_i - r_i(X^TX)^{-1}x_ih_{ii}Y_i \\ &= \hat\beta + r_i(X^TX)^{-1}x_ix_i^T\hat\beta - (r_ih_{ii}+1)(X^TX)^{-1}x_iY_i \\ &= \hat\beta - r_i(X^TX)^{-1}x_i(Y_i - x_i^T\hat\beta) \\ &= \hat\beta - r_i(X^TX)^{-1}x_i\hat\epsilon_i\end{aligned}$$

(g) Deduce that $\hat\mu_{(i)} = \hat\mu_i - r_ih_{ii}\hat\epsilon_i$.
Multiply $\hat\beta_{(i)}$ by x_i to get

$$\hat\mu_{(i)} = \hat\mu - r_ix_i(X^TX)^{-1}x_i\hat\epsilon_i = \hat\mu - r_ih_{ii}\hat\epsilon_i$$

(h) Show that the i^{th} PRESS residual $Y_i - \hat\mu_{(i)}$ is given by

$$Y_i - \hat\mu_{(i)} = r_i\hat\epsilon_i$$

$$PRESS_i = Y_i - \hat\mu_{(i)} = Y_i - \hat\mu_i + r_ih_{ii}\hat\epsilon_i = (1 + r_ih_{ii})\hat\epsilon_i = r_i\hat\epsilon_i$$

(i) Derive the formula for the i^{th} externally studentized (case deleted) residual.
The variance of the PRESS residual is $Var(r_i\hat\epsilon_i) = \sigma^2r_i^2(1-h_{ii}) = \sigma^2r_i$. The externally studentized residual is the PRESS residual divided by an estimate of its standard error where σ^2 is estimated by $MSE_{(i)}$. Thus the externally studentized residual is

$$\frac{PRESS_i}{\sqrt{MSE_{(i)}r_i}} = \frac{\hat\epsilon_i\sqrt{r_i}}{\sqrt{MSE_{(i)}}} = \frac{\hat\epsilon_i}{\sqrt{MSE_{(i)}(1-h_{ii})}}$$

To get the simplest formula we must actually simplify $MSE_{(i)}$.

$$\begin{aligned}MSE_{(i)} &= \frac{\sum_{j\neq i}(Y_j - x_j^T\hat\beta_{(i)})^2}{(n-1)-p} \\ &= \frac{\sum_j(Y_j - x_j^T(\hat\beta - r_i(X^TX)^{-1}x_i\hat\epsilon_i))^2 - PRESS_i^2}{n-p-1}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sum(Y_j - x_j^T\hat\beta)^2}{n-p-1} + \frac{2r_i\sum(Y_j - x_j^T\hat\beta)x_j^T(X^TX)^{-1}x_i}{n-p-1} \\
&\quad + \frac{r_i^2 x_i^T(X^TX)^{-1}\sum x_j x_j^T(X^TX)^{-1}x_i\hat\epsilon_i^2 - r_i^2\hat\epsilon_i^2}{n-p-1} \\
&= \frac{ESS + 0 + r_i^2 x_i^t(X^TX)^{-1}x_i\hat\epsilon_i^2 - r_i^2\hat\epsilon_i^2}{n-p-1}
\end{aligned}
$$

The middle term vanishes because

$$\sum(Y_j - x_j^T\hat\beta)x_j = X^TY - X^TX\hat\beta = 0\,.$$

Use $r_i^2(1-h_{ii}) = r_i$ to get

$$MSE_{(i)} = \frac{ESS - r_i\hat\epsilon_i^2}{n-p-1}$$

and then you can deduce the final formula given in class.

2. For the Nitrogen Output in Wallabies data set from Assignment 3 do forward, backward, stepwise and all subsets regression.

   Here is code for all the methods and with all subsets done both using C_p and using adjusted R^2.

```
data nit;
  infile 'nit.dat' ;
  input nitexc weight dryin wetin nitin ;
proc reg  data=nit;
   model nitexc = weight dryin wetin
         nitin /selection=FORWARD;
run ;
proc reg  data=nit;
   model nitexc = weight dryin wetin
         nitin /selection=BACKWARD;
run ;
proc reg  data=nit;
   model nitexc = weight dryin wetin
         nitin /selection=STEPWISE;
run ;
proc reg  data=nit;
   model nitexc = weight dryin wetin
         nitin /selection=CP;
run ;
proc reg  data=nit;
   model nitexc = weight dryin wetin
         nitin /selection=ADJRSQ;
run ;
```

The conclusion of the output is that BACKWARD and STEPWISE settle for the model containing only Nitrogen Intake as a predictor. The forward selection method also includes Wet Intake because of the very high level of α (0.5) to enter. The all subsets method using C_p would settle on the model using only nitrogen intake but the adjusted R^2 method also includes Wet Intake. However, overall there seems little reason to include Wet Intake since it improves the fit very little and is not very significant at all.

3. Problem 23.12, 23.13 parts c,d,e,f,g only and 23.14.

   Here is code for the analysis of variance.

```
options pagesize=60 linesize=80;
data electron;
 infile 'anova.dat' firstobs=2;
 input Time Sex Sequence Exper Replic;
proc glm  data=electron;
 class Sex Sequence Exper ;
 model time = Sex|Sequence|Exper ;
 means sex sequence exper sex*sequence*exper;
 estimate 'sexdif' sex 1 -1;
 estimate 'seq12dif' sequence 1 -1 0;
 estimate 'seq13dif' sequence 1 0 -1;
 estimate 'seq23dif' sequence 0 1 -1;
 estimate 'expdif' exper 1 -1;
 output out=anovres r=resid;
proc rank data=anovres normal=blom out=ressc;
 var resid;
 ranks nscores;
proc corr data=ressc;
 var resid nscores;
run;
```

   COMMENTS

   - You use the `class` statement to let SAS know that the factors are categorical.
   - The `model` statement requests that all two way and the three way interaction terms be fitted.
   - The `means` statement computes means and standard deviations for the groups defined by the levels of sex, of sequence, of exper and of all the 3 way combinations.
   - The `estimate` statements compute the differences in means between the two sexes, between the 1st and second sequence, the first and third sequence and between the two levels of experience. This hinges on the use of a balanced design.
   - The `proc rank` statements compute approximations to the expected values of normal order statistics needed for the correlation test in 23.12b.

- The correlation is actually computed by `proc corr`.

You can read the type III sums of squares table to do F tests without doing multiple runs because each effect has a sum of squares which is unaffected by the presence of the others. The conclusion is that the three way interaction is insignificant and all three two way interactions are insignificant. All three main effects are significant; none can be eliminated.

In the question on Bonferroni confidence intervals the 5 quantities to be estimated are each estimated by a difference of two averages so that the variance of the estimated difference is of the form

$$\sigma^2(\frac{1}{n_1} + \frac{1}{n_2})$$

You work out standard t type confidence intervals by estimating the means as usual (see the `means` statment) and replacing σ^2 by the MSE in the formula for the variance. The Bonferroni method just replaces the α of 0.01 by $\alpha/5 = 0.02$ for 5 confidence intervals. The t critical value is $t_{0.02/2,48} = 2.41$. For the sex difference for instance the average time for men is 1155.933 seconds while for women it is 966.1333. The difference is 189.8 and this is computed by the means statement whose output is

```
                          T for H0:    Pr>|T| Std Error of
Parameter     Estimate Parameter=0           Estimate
sexdif     189.800000     25.10      0.0001  7.56325180
seq12dif   -57.250000     -6.18      0.0001  9.26305385
seq13dif     6.600000      0.71      0.4796  9.26305385
seq23dif    63.850000      6.89      0.0001  9.26305385
expdif     159.666667     21.11      0.0001  7.56325180
```

Notice the standard errors. The mean squared error in the model is 858.06. There are 30 men and 30 women so the variance of the difference between men's average and women's average is $858.06(1/30+1/30) = 57.204$. The standard error of the estimate of $\mu_{1\cdot\cdot} - \mu_{2\cdot\cdot}$ is then $\sqrt{57.204} = 7.563$ as in the output. The desired confidence intervals are all and estimate from the column labelled `Estimate` plus or minus 2.41 times a standard error from the last column.

The needed mean for $\hat{\mu}_{231}$ is produced by `means`. The key point is that the standard error to attach to the mean is $\sqrt{MSE/5}$ which is based on 48 degrees of freedom, not on 4.

4. Problem 25.11 parts a, b anc c, 25.12 part c.

   For this question I used the SAS code

```
data knees;
  infile 'knees.dat' firstobs=2;
  input Days Fitness Patient Age ;
proc glm  data=knees;
```

```
  class Fitness ;
  model Days = Fitness|Age ;
proc glm  data=knees;
  class Fitness ;
  model Days = Fitness Age ;
  output out=anovres r=resid p=fitted;
proc rank data=anovres normal=blom out=ressc;
  var resid;
  ranks nscores;
proc corr data=ressc;
  var resid nscores;
proc print data=ressc;
  var fitted fitness age resid nscores;
run;
```

getting the output

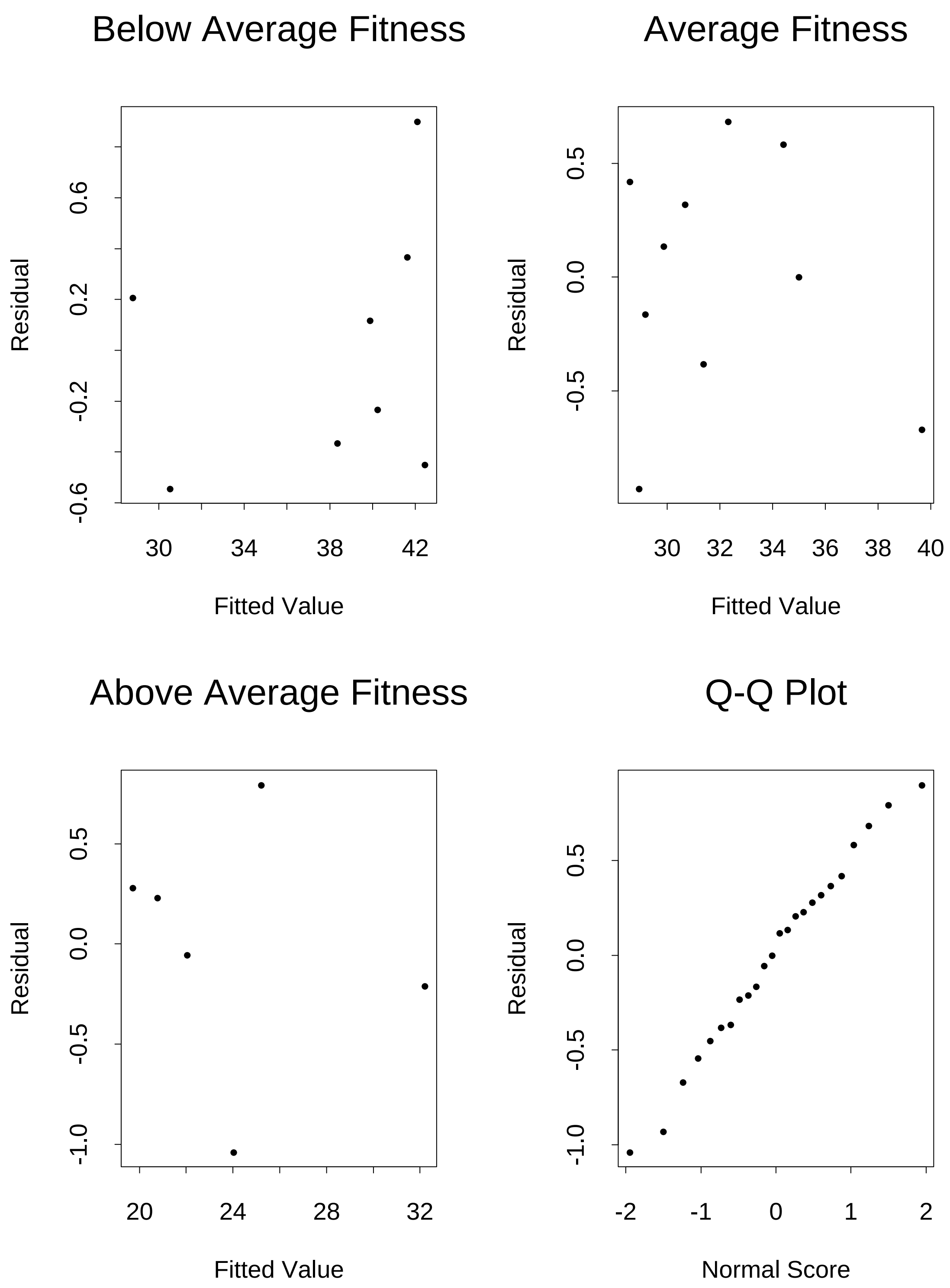
```
                        General Linear Models Procedure
Dependent Variable: DAYS
                       Sum of         Mean
Source         DF      Squares      Square F Value Pr > F
Model           5 1082.0560870 216.4112174  655.36 0.0001
Error          18    5.9439130   0.3302174
Correctd Totl 23 1088.0000000
        R-Square         C.V.      Root MSE  DAYS Mean
        0.994537     1.795767     0.5746454  32.000000
Source       DF  Type III SS   Mean Square F Value Pr > F
FITNESS       2   5.44989183    2.72494592    8.25 0.0029
AGE           1 369.44147783  369.44147783 1118.78 0.0001
AGE*FITNESS  2   0.22183487    0.11091744    0.34 0.7191

                        General Linear Models Procedure
Dependent Variable: DAYS
                       Sum of         Mean
Source         DF      Squares      Square F Value Pr > F
Model           3 1081.8342521 360.6114174 1169.72 0.0001
Error          20    6.1657479   0.3082874
Correctd Totl 23 1088.0000000
    R-Square         C.V.      Root MSE      DAYS Mean
    0.994333     1.735114     0.5552363      32.000000
Source  DF  Type III SS   Mean Square F Value   Pr > F
FITNESS  2 246.08370505  123.04185252  399.11   0.0001
AGE      1 409.83425209  409.83425209 1329.39   0.0001
```

Correlation Analysis
Pearson Correlation Coefficients

	RESID
NSCORES	0.99488

OBS	FITTED	FITNESS	AGE	RESID	NSCORES
1	28.7930	1	18.3	0.20697	0.26136
2	42.4503	1	30.0	-0.45028	-0.87524
3	38.3648	1	26.5	-0.36478	-0.60318
4	40.2324	1	28.1	-0.23244	-0.48332
5	42.1001	1	29.7	0.89991	1.94690
6	39.8822	1	27.8	0.11775	0.05171
7	30.5440	1	19.8	-0.54396	-1.03865
8	41.6332	1	29.3	0.36682	0.73241
9	29.8639	2	20.8	0.13613	0.15568
10	34.9999	2	25.2	0.00007	-0.05171
11	39.6691	2	29.2	-0.66907	-1.23590
12	28.9300	2	20.0	-0.93004	-1.49843
13	30.6810	2	21.5	0.31903	0.60318
14	31.3813	2	22.1	-0.38134	-0.73241
15	28.5799	2	19.7	0.42015	0.87524
16	34.4163	2	24.7	0.58372	1.03865
17	29.1635	2	20.2	-0.16349	-0.26136
18	32.3152	2	22.9	0.68483	1.23590
19	25.2062	3	22.7	0.79380	1.49843
20	32.2099	3	28.7	-0.20991	-0.37006
21	20.7705	3	18.9	0.22949	0.37006
22	19.7199	3	18.0	0.28005	0.48332
23	24.0389	3	21.7	-1.03891	-1.94690
24	22.0545	3	20.0	-0.05452	-0.15568

For part a of 25.11 the residuals are printed out above. For part b the plots desired are:



The conclusion is that neither the residual plots nor the Q-Q plot show any major problems.

For part c, the generalized model is

$$Y_{ij} = \mu + \tau_i + \gamma_i(x_{ij} - \bar{x}_{..}) + \epsilon_{ij}$$

and the null hypothesis is that all the γ_i are equal. This is tested by comparing the two model statements `model days = fitness | age` and `model days = fitness age`, doing an extra sum of squares F test. The resulting F statistic is the Type III sum of squares for fitness*age giving $F = 0.34$ and $P = 0.7191$. The null hypothesis is accepted.

For 25.12 part c the F statistic is obtained from the Type III sum of squares for fitness for the `model days = fitness age` statement. This has $F = 399.11$ and $P = 0.0001$ There is clearly an effect of the variable Fitness.

The `estimate` statements permit us to compare the three levels of fitness. SAS prints out estimates of the differences in the intercepts. The relevant output is

```
                              T for H0: Pr>|T| Std Err of
Parameter           Estimate  Param=0            Estimate
High v Low Fitness -8.72289277 -26.20   0.0001 0.33296397
High v Med Fitness -6.87551411 -23.84   0.0001 0.28837673
Med  v Low Fitness -1.84737866  -6.44   0.0001 0.28694289
```

Notice that the fit group appears to recuperate about 9 days faster than the unfit group!