STAT 380 Lecture 10

Reading for Today's Lecture: Chapter 4 sections 1-3.
Goals of Today's Lecture:

- Define communicating classes
- Define recurrent, transient, periodic states.

Classification of States

State i **leads to** state j if $\mathbf{P}^n_{ij} > 0$ for some n. It is convenient to agree that $\mathbf{P}^0 = \mathbf{I}$ the identity matrix; thus i leads to i.

Note i leads to j and j leads to k implies i leads to k (Chapman-Kolmogorov).

States i and j **communicate** if i leads to j and j leads to i.

The relation of communication is an **equivalence** relation; it is reflexive, symmetric and transitive: if i and j communicate and j and k communicate then i and k communicate.

Example (+ signs indicate non-zero entries):

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ + & + & + & 0 & 0 \\ + & 0 & 0 & + & 0 \\ 0 & + & 0 & + & + \end{bmatrix}$$

For this example:
$1 \rightsquigarrow 2$, $2 \rightsquigarrow 3$, $3 \rightsquigarrow 1$ so $1, 2, 3$ are all in the same communicating class.
$4 \rightsquigarrow 1, 2, 3$ but not vice versa.
$5 \rightsquigarrow 1, 2, 3, 4$ but not vice versa.
So the communicating classes are

$$\{1, 2, 3\} \quad \{4\} \quad \{5\}$$

A Markov Chain is **irreducible** if there is only one communicating class.
Notation:

$$f_i = P(\exists n > 0 : X_n = i | X_0 = i)$$

State i is **recurrent** if $f_i = 1$, otherwise **transient**.

If $f_i = 1$ then Markov property (chain starts over when it gets back to i) means prob return infinitely many times (given started in i or given ever get to i) is 1.

Consider chain started from transient i. Let N be number of visits to state i (including visit at time 0). To return $m-1$ times must return once then starting over return $m-1$ times, then never return. So:

$$P(N = m) = f_i^{m-1}(1 - f_i)$$

for $m = 1, 2, \ldots$.

N has a Geometric distribution and $\mathrm{E}(N) = 1/(1 - f_i)$.

Another calculation:

$$N = \sum_0^\infty 1(X_k = i)$$

so

$$\mathrm{E}(N) = \sum_0^\infty P(X_k = i)$$

If we start the chain in state i then this is

$$\mathrm{E}(N) = \sum_0^\infty \mathbf{P}_{ii}^n$$

and i is transient if and only if

$$\sum_0^\infty \mathbf{P}_{ii}^n < \infty\,.$$

For last example: 4 and 5 are transient. Claim: states 1, 2 and 3 are recurrent.

Proof: argument above shows each transient state is visited only finitely many times. So: there is a recurrent state. (Note use of finite number of states.) It must be one of 1, 2 and 3. **Proposition**: If one state in a communicating class is recurrent then all states in the communicating class are recurrent.

Proof: Let i be the known recurrent state so

$$\sum_n \mathbf{P}_{ii}^n = \infty$$

Assume i and j communicate. Find integers m and k such that

$$\mathbf{P}_{ij}^m > 0$$

and

$$\mathbf{P}_{ji}^k > 0$$

Then

$$\mathbf{P}_{jj}^{m+n+k} \ge \mathbf{P}_{ji}^k \mathbf{P}_{ii}^n \mathbf{P}_{ij}^m$$

Sum RHS over n get ∞ so

$$\sum_n \mathbf{P}_{jj}^n = \infty$$

Proposition also means that if 1 state in a class is transient so are all.

State i has period d if d is greatest common divisor of

$$\{n : \mathbf{P}_{ii}^n > 0\}$$

If i and j are in the same class then i and j have same period. If $d = 1$ then state i is called **aperiodic**; if $d > 1$ then i is periodic.

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For this example $\{1, 2, 3\}$ is a class of period 3 states and $\{4, 5\}$ a class of period 2 states.

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

has a single communicating class of period 2.

A chain is **aperiodic** if all its states are aperiodic.

Infinite State Spaces

Example: consider a sequence of independent coin tosses with probability p of Heads on a single toss. Let X_n be number of heads minus number of tails after n tosses. Put $X_0 = 0$.

X_n is a Markov Chain. State space is $\mathbb{Z}$, the integers and

$$\mathbf{P}_{ij} = \begin{cases} p & j = i + 1 \\ 1 - p & j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

This chain has one communicating class (for $p \neq 0, 1$) and all states have period 2. According to the strong law of large numbers X_n/n converges to $2p-1$. If $p \neq 1/2$ this guarantees that for all large enough n $X_n \neq 0$, that is, the number of returns to 0 is not infinite. The state 0 is then transient and so **all** states must be transient.

For $p = 1/2$ the situation is different. It is a fact that

$$\mathbf{P}_{00}^n = P(\#\ \mathrm{H} = \#\ \mathrm{T} \text{ at time } n)$$

For n even this is the probability of exactly $n/2$ heads in n tosses.

Local Central Limit Theorem (normal approximation to $P(-1/2 < X_n < 1/2)$) (or Stirling's approximation) shows

$$\sqrt{2m}P(\text{Binomial}(2m, 1/2) = m) \to (2\pi)^{-1/2}$$

so:

$$\sum_n \mathbf{P}_{00}^n = \infty$$

That is: 0 is a recurrent state.