STAT 380 Lecture 11

Hitting Times

Start irreducible recurrent chain X_n in state i. Let T_j be first $n > 0$ such that $X_n = j$. Define

$$m_{ij} = \mathrm{E}(T_j | X_0 = i)$$

First step analysis:

$$\begin{aligned} m_{ij} &= 1 \cdot P(X_1 = j | X_0 = i) \\ &\qquad + \sum_{k \neq j} (1 + \mathrm{E}(T_j | X_0 = k)) P_{ik} \\ &= \sum_j P_{ij} + \sum_{k \neq j} P_{ik} m_{kj} \\ &= 1 + \sum_{k \neq j} P_{ik} m_{kj} \end{aligned}$$

Example

$$\mathbf{P} = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

The equations are

$$\begin{aligned} m_{11} &= 1 + \frac{2}{5} m_{21} \\ m_{12} &= 1 + \frac{3}{5} m_{12} \\ m_{21} &= 1 + \frac{4}{5} m_{21} \\ m_{22} &= 1 + \frac{1}{5} m_{12} \end{aligned}$$

The second and third equations give immediately

$$\begin{aligned} m_{12} &= \frac{5}{2} \\ m_{21} &= 5 \end{aligned}$$

Then plug in to the others to get

$$\begin{aligned} m_{11} &= 3 \\ m_{22} &= \frac{3}{2} \end{aligned}$$

Notice stationary initial distribution is

$$\left(\frac{1}{m_{11}}, \frac{1}{m_{22}} \right)$$

Consider fraction of time spent in state j:

$$\frac{1(X_0 = j) + \cdots + 1(X_n = j)}{n+1}$$

Imagine chain starts in chain i; take expected value.

$$\frac{\sum_{r=1}^n \mathbf{P}_{ij}^r + 1(i = j)}{n+1}$$

If rows of $\mathbf{P}$ converge to π then fraction converges to π_j; i.e. limiting fraction of time in state j is π_j.

Heuristic: start chain in i. Expect to return to i every m_{ii} time units. So are in state i about once every m_{ii} time units; i.e. limiting fraction of time in state i is $1/m_{ii}$.

Conclusion: for an irreducible recurrent finite state space Markov chain

$$\pi_i = \frac{1}{m_{ii}}.$$

Infinite State Spaces

Previous conclusion is still right **if** there is a stationary initial distribution.

Example: $X_n = \text{Heads} - \text{Tails}$ after n tosses of fair coin. Equations are

$$\begin{aligned} m_{0,0} &= 1 + \frac{1}{2} m_{1,0} + \frac{1}{2} m_{-1,0} \\ m_{1,0} &= 1 + \frac{1}{2} m_{2,0} \end{aligned}$$

and many more.

Some observations:

You have to go through 1 to get to 0 from 2 so

$$m_{2,0} = m_{2,1} + m_{1,0}$$

Symmetry (switching H and T):

$$m_{1,0} = m_{-1,0}$$

The transition probabilities are **homogeneous**:

$$m_{2,1} = m_{1,0}$$

Conclusion:

$$\begin{aligned} m_{0,0} &= 1 + m_{1,0} \\ &= 1 + 1 + \frac{1}{2} m_{2,0} \\ &= 2 + m_{1,0} \end{aligned}$$

Notice that there are **no** finite solutions!

Summary of the situation:

Every state is recurrent.

All the expected hitting times m_{ij} are infinite.

All entries $\mathbf{P}_{ij}^n$ converge to 0.

Jargon: The states in this chain are null recurrent.