STAT 380 Lecture 12

One Example

Page 229, question 21: runner goes from front or back door, prob 1/2 each. Returns front or back, prob 1/2 each. Has k pairs of shoes, wears pair if any at departure door, leaves at return door. No shoes? Barefoot. Long run fraction of time barefoot?

Solution: Let X_n be number of shoes at front door on day n. Then X_n is a Markov Chain.

Transition probabilities:

k pairs at front door on day n. X_{n+1} is k if goes out back door (prob is 1/2) or out front door and back in front door (prob is 1/4). Otherwise X_{n+1} is $k-1$.

$0 < j < k$ pairs at front door on day n. X_{n+1} is $j+1$ if out back, in front (prob is 1/4). X_{n+1} is $j-1$ if out front, in back. Otherwise X_{n+1} is j.

0 pairs at front door on day n. X_{n+1} is 0 if out front door (prob 1/2) or out back door and in back door (prob 1/4) otherwise X_{n+1} is 1.

Transition matrix $\mathbf{P}$:

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Doubly stochastic: row sums and column sums are 1.

So $\pi_i = 1/(k+1)$ for all i is stationary initial distribution.

Solution to problem: 1 day in $k+1$ no shoes at front door. Half of those go barefoot. Also 1 day in $k+1$ all shoes at front door; go barefoot half of these days. Overall go barefoot $1/(k+1)$ of the time.

Gambler's Ruin

Insurance company's reserves fluctuate: sometimes up, sometimes down. Ruin is event they hit 0 (company goes bankrupt). General problem. For given model of fluctuation compute probability of ruin either eventually or in next k time units.

Simplest model: gambling on Red at Casino. Bet \$1 at a time. Win \$1 with probability p, lose \$1 with probability $1-p$. Start with k dollars. Quit playing when down to \$0 or up to N. Compute

$$P_k = P(\text{reach } N \text{ before } 0)$$

X_n = fortune after n plays. $X_0 = k$.

Transition matrix:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1-p & 0 & p & 0 & \cdots & 0 \\ 0 & 1-p & 0 & p & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

First step analysis:

$$\begin{aligned} P_0 &= 0 \\ P_i &= (1-p)P_{i-1} + pP_{i+1} \\ P_N &= 1 \end{aligned}$$

Middle equation is

$$pP_i + (1-p)P_i = (1-p)P_{i-1} + pP_{i+1}$$

or

$$\begin{aligned} P_{i+1} - P_i &= \frac{1-p}{p}(P_i - P_{i-1}) \\ &= \left(\frac{1-p}{p}\right)^2 (P_{i-1} - P_{i-2}) \\ &\vdots \\ &= \left(\frac{1-p}{p}\right)^i (P_1 - P_0) \\ &= \left(\frac{1-p}{p}\right)^i P_1 \end{aligned}$$

Sum from $i = 0$ to $i = k-1$ to get

$$P_k = \sum_{i=0}^{k-1} \left(\frac{1-p}{p}\right)^i P_1$$

or

$$P_k = \frac{1 - \{(1-p)/p\}^k}{1 - \{(1-p)/p\}} P_1$$

For $k = N$ we get

$$1 = \frac{1 - \{(1-p)/p\}^N}{1 - \{(1-p)/p\}} P_1$$

so that

$$P_k = \frac{1 - \{(1-p)/p\}^k}{1 - \{(1-p)/p\}^N} P_1$$

Notice that if $p = 1/2$ our formulas for the sum of the geometric series are wrong. But for $p = 1/2$ we get

$$P_k = kP_1$$

so

$$P_k = \frac{k}{N}.$$