STAT 380 Lecture 13

Mean time in transient states

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

States 3 and 4 are transient. Let $m_{i,j}$ be the expected total number of visits to state j for chain started in i.

For $i = 1$ or $i = 2$ and $j = 3$ or 4:

$$m_{ij} = 0$$

For $j = 1$ or $j = 2$

$$m_{ij} = \infty$$

For $i, j \in \{3, 4\}$ first step analysis:

$$\begin{aligned} m_{3,3} &= 1 + \frac{1}{4}m_{3,3} + \frac{1}{4}m_{4,3} \\ m_{3,4} &= 0 + \frac{1}{4}m_{3,4} + \frac{1}{4}m_{4,4} \\ m_{4,3} &= 0 + \frac{3}{8}m_{3,3} + \frac{1}{8}m_{4,3} \\ m_{4,4} &= 0 + \frac{3}{8}m_{3,4} + \frac{1}{8}m_{4,4} \end{aligned}$$

In matrix form

$$\begin{bmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} m_{3,3} & m_{3,4} \\ m_{4,3} & m_{4,4} \end{bmatrix}$$

Translate to matrix notation:

$$\mathbf{M} = \mathbf{I} + \mathbf{P}_T\mathbf{M}$$

where $\mathbf{I}$ is the identity, $\mathbf{M}$ is the matrix of means and $\mathbf{P}_T$ the part of the transition matrix corresponding to transient states.

Solution is

$$\mathbf{M} = (\mathbf{I} - \mathbf{P}_T)^{-1}$$

In our case

$$\mathbf{I} - \mathbf{P}_T = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \\ -\frac{3}{8} & \frac{7}{8} \end{bmatrix}$$

so that

$$\mathbf{M} = \begin{bmatrix} \frac{14}{9} & -\frac{4}{9} \\ \\ -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

Poisson Processes

Particles arriving over time at a particle detector. Several ways to describe most common model.

Approach 1: numbers of particles arriving in an interval has Poisson distribution, mean proportional to length of interval, numbers in several non-overlapping intervals independent.

For $s < t$, denote number of arrivals in $(s, t]$ by $N(s, t)$. Model is

1. $N(s, t)$ has a Poisson$(\lambda(t - s))$ distribution.
2. For $0 \le s_1 < t_1 \le s_2 < t_2 \cdots \le s_k < t_k$ the variables $N(s_i, t_i); i = 1, \ldots, k$ are independent.

Approach 2:

Let $0 < S_1 < S_2 < \cdots$ be the times at which the particles arrive.

Let $T_i = S_i - S_{i-1}$ with $S_0 = 0$ by convention.

Then $T_1, T_2, \ldots$ are independent Exponential random variables with mean $1/\lambda$.

Note $P(T_i > x) = e^{-\lambda x}$ is called **survival function** of T_i.

Approaches are equivalent. Both are deductions of a model based on **local** behaviour of process.

Approach 3: Assume:

1. given all the points in $[0, t]$ the probability of 1 point in the interval $(t, t + h]$ is of the form
$$\lambda h + o(h)$$
2. given all the points in $[0, t]$ the probability of 2 or more points in interval $(t, t + h]$ is of the form
$$o(h)$$

All 3 approaches are equivalent. I show: 3 implies 1, 1 implies 2 and 2 implies 3. First explain o, O.

Notation: given functions f and g we write

$$f(h) = g(h) + o(h)$$

provided

$$\lim_{h \to 0} \frac{f(h) - g(h)}{h} = 0$$

[Aside: if there is a constant M such that

$$\limsup_{h \to 0} \left| \frac{f(h) - g(h)}{h} \right| \leq M$$

we say

$$f(h) = g(h) + O(h)$$

Notation due to Landau. Another form is

$$f(h) = g(h) + O(h)$$

means there is $\delta > 0$ and M s.t. for all $|h| < \delta$

$$|f(h) - g(h)| \leq Mh$$

Idea: $o(h)$ is tiny compared to h while $O(h)$ is (very) roughly the same size as h.]