STAT 380: Lecture 14

Equivalence of the models

Model 3 implies 1: Fix t, define $f_t(s)$ to be conditional probability of 0 points in $(t, t+s]$ given value of process on $[0, t]$.

Derive differential equation for f. Given process on $[0, t]$ and 0 points in $(t, t+s]$ probability of no points in $(t, t+s+h]$ is

$$f_{t+s}(h) = 1 - \lambda h + o(h)$$

Given the process on $[0, t]$ the probability of no points in $(t, t+s]$ is $f_t(s)$. Using $P(AB|C) = P(A|BC)P(B|C)$ gives

$$\begin{aligned} f_t(s+h) &= f_t(s) f_{t+s}(h) \\ &= f_t(s)(1 - \lambda h + o(h)) \end{aligned}$$

Now rearrange, divide by h to get

$$\frac{f_t(s+h) - f_t(s)}{h} = \lambda f_t(s) + \frac{o(h)}{h}$$

Let $h \to 0$ and find

$$\frac{\partial f_t(s)}{\partial s} = -\lambda f_t(s)$$

Differential equation has solution

$$f_t(s) = f_t(0) \exp(-\lambda s) = \exp(-\lambda s) .$$

Notice: survival function of exponential rv..

General case:

Notation: $N(t) = N(0, t)$.

$N(t)$ is a non-decreasing function of t. Let

$$P_k(t) = P(N(t) = k)$$

Evaluate $P_k(t+h)$ by conditioning on $N(s); 0 \le s < t$ and $N(t) = j$.

Given $N(t) = j$ probability that $N(t+h) = k$ is conditional probability of $k - j$ points in $(t, t+h]$.

So, for $j \le k - 2$:

$$P(N(t+h) = k | N(t) = j, N(s), 0 \le s < t) = o(h)$$

For $j = k-1$ we have

$$P(N(t+h) = k | N(t) = k-1, N(s), 0 \le s < t) = \lambda h + o(h)$$

For $j = k$ we have

$$P(N(t+h) = k | N(t) = k, N(s), 0 \le s < t) = 1 - \lambda h + o(h)$$

N is increasing so only consider $j \le k$.

$$\begin{aligned} P_k(t+h) &= \sum_{j=0}^{k} P(N(t+h) = k | N(t) = j) P_j(t) \\ &= P_k(t)(1-\lambda h) + \lambda h P_{k-1}(t) + o(h) \end{aligned}$$

Rearrange, divide by h and let $h \to 0$ t get

$$P_k'(t) = -\lambda P_k(t) + \lambda P_{k-1}(t)$$

For $k = 0$ the term P_{k-1} is dropped and

$$P_0'(t) = \lambda P_0(t)$$

Using $P_0(0) = 1$ we get

$$P_0(t) = e^{-\lambda t}$$

Put this into the equation for $k = 1$ to get

$$P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

Multiply by $e^{\lambda t}$ to see

$$\left(e^{\lambda t} P_1(t)\right)' = \lambda$$

With $P_1(0) = 0$ we get

$$P_1(t) = \lambda t e^{-\lambda t}$$

For general k we have $P_k(0) = 0$ and

$$\left(e^{\lambda t} P_k(t)\right)' = \lambda e^{\lambda t} P_{k-1}(t)$$

Check by induction that

$$e^{\lambda t} P_k(t) = (\lambda t)^k / k!$$

Hence: $N(t)$ has Poisson(λt) distribution.

Similar ideas permit proof of

$$P(N(s,t) = k | N(u); 0 \le u \le s) = \frac{\{\lambda(t-s)\}^k e^{-\lambda}}{k!}$$

From which (by induction) we can prove that N has independent Poisson increments.

Exponential Interarrival Times

If N is a Poisson Process we define $T_1, T_2, \ldots$ to be the times between 0 and the first point, the first point and the second and so on.

Fact: $T_1, T_2, \ldots$ are iid exponential rvs with mean $1/\lambda$.

We already did T_1 rigorously. The event $T > t$ is exactly the event $N(t) = 0$. So

$$P(T > t) = \exp(-\lambda t)$$

which is the survival function of an exponential rv.

I do case of T_1, T_2. Let t_1, t_2 be two positive numbers and $s_1 = t_1$, $s_2 = t_1 + t_2$. The event

$$\{t_1 < T_1 \le t_1 + \delta_1\} \cap \{t_2 < T_2 \le t_2 + \delta_2\} .$$

This is almost the same as the intersection of the four events:

$$\begin{aligned} N(0, t_1] &= 0 \\ N(t_1, t_1 + \delta_1] &= 1 \\ N(t_1 + \delta_1, t_1 + \delta_1 + t_2] &= 0 \\ N(s_2 + \delta_1, s_2 + \delta_1 + \delta_2] &= 1 \end{aligned}$$

which has probability

$$e^{-\lambda t_1} \times \lambda \delta_1 e^{-\lambda \delta_1} \times e^{-\lambda t_2} \times \lambda \delta_2 e^{-\lambda \delta_2}$$

Divide by $\delta_1 \delta_2$ and let δ_1 and δ_2 go to 0 to get joint density of T_1, T_2 is

$$e^{-\lambda t_1} e^{-\lambda t_2}$$

which is the joint density of two independent exponential variates.

More rigor:

- Find joint density of $S_1, \ldots, S_k$.
- Use **change of variables** to find joint density of $T_1, \ldots, T_k$.

First step: Compute

$$P(0 < S_1 \le s_1 < S_2 \le s_2 \cdots < S_k \le s_k)$$

This is just the event of exactly 1 point in each interval $(s_{i-1}, s_i]$ for $i = 1, \ldots, k-1$ ($s_0 = 0$) and at least one point in $(s_{k-1}, s_k]$ which has probability

$$\prod_1^{k-1} \left\{ \lambda(s_i - s_{i-1}) e^{-\lambda(s_i - s_{i-1})} \right\} \left(1 - e^{-\lambda(s_k - s_{k-1})}\right)$$

Second step: write this in terms of joint cdf of $S_1, \ldots, S_k$. I do $k = 2$:

$$P(0 < S_1 \le s_1 < S_2 \le s_2) = F_{S_1,S_2}(s_1, s_2) - F_{S_1,S_2}(s_1, s_1)$$

Notice tacit assumption $s_1 < s_2$.

Differentiate twice, that is, take

$$\frac{\partial^2}{\partial s_1 \partial s_2}$$

to get

$$f_{S_1,S_2}(s_1, s_2) = \frac{\partial^2}{\partial s_1 \partial s_2} \lambda s_1 e^{-\lambda s_1} \left(1 - e^{-\lambda(s_2 - s_1)}\right)$$

Simplify to

$$\lambda^2 e^{-\lambda s_2}$$

Recall tacit assumption to get

$$f_{S_1,S_2}(s_1, s_2) = \lambda^2 e^{-\lambda s_2} 1(0 < s_1 < s_2)$$

That completes the first part.

Now compute the joint cdf of T_1, T_2 by

$$F_{T_1,T_2}(t_1, t_2) = P(S_1 < t_1, S_2 - S_1 < t_2)$$

This is

$$
\begin{aligned}
P(S_1 < t_1, &S_2 - S_1 < t_2) \\
&= \int_0^{t_1} \int_{s_1}^{s_1+t_2} \lambda^2 e^{-\lambda s_2} ds_2 ds_1 \\
&= \lambda \int_0^{t_1} \left(e^{-\lambda s_1} - e^{-\lambda(s_1+t_2)}\right) ds_1 \\
&= 1 - e^{-\lambda t_1} - e^{-\lambda t_2} + e^{-\lambda(t_1+t_2)}
\end{aligned}
$$

Differentiate twice to get

$$f_{T_1,T_2}(t_1, t_2) = \lambda e^{-\lambda t_1} \lambda e^{-\lambda t_2}$$

which is the joint density of two independent exponential random variables.

Summary so far:

Have shown:

Instantaneous rates model implies independent Poisson increments model implies independent exponential interarrivals.

Next: show independent exponential interarrivals implies the instantaneous rates model.

Suppose $T_1, \ldots$ iid exponential rvs with means $1/\lambda$. Define N_t by $N_t = k$ if and only if

$$T_1 + \cdots + T_k \le t \le T_1 + \cdots + T_{k+1}$$

Let A be the event $N(s) = n(s); 0 < s \le t$. We are to show

$$P(N(t, t+h] = 1 | N(t) = k, A) = \lambda h + o(h)$$

and

$$P(N(t, t+h] \ge 2 | N(t) = k, A) = o(h)$$

If $n(s)$ is a possible trajectory consistent with $N(t) = k$ then n has jumps at points

$$t_1, s_2 \equiv t_1 + t_2, \ldots, s_k \equiv t_1 + \cdots + t_k < t$$

and at no other points in $(0, t]$.

So given $N(s) = n(s); 0 < s \le t$ with $n(t) = k$ we are essentially being given

$$T_1 = t_1, \ldots, T_k = t_k, T_{k+1} > t - s_k$$

and asked the conditional probabilty in the first case of the event B given by

$$t - s_k < T_{k+1} \leq t - s_k + h < T_{k+2} + T_{k+1} \, .$$

Conditioning on $T_1, \ldots, T_k$ irrelevant (independence).

$$\begin{aligned} P(N(t, t+h] = 1 | N(t) = k, A)/h & \\ &= P(B | T_{k+1} > t - t_k)/h \\ &= \frac{P(B)}{h e^{-\lambda(t-t_k)}} \end{aligned}$$

The numerator may be evaluated by integration:

$$P(B) = \int_{t-s_k}^{t-s_k+h} \int_{t-s_k+h-u_1}^{\infty} \lambda^2 e^{-\lambda(u_1+u_2)} du_2 du_1$$

Let $h \to 0$ to get the limit

$$P(N(t, t+h] = 1 | N(t) = k, A)/h \to \lambda$$

as required.

The computation of

$$\lim_{h \to 0} P(N(t, t+h] \geq 2 | N(t) = k, A)/h$$

is similar.

Properties of exponential rvs

Convolution: If X and Y independent rvs with densities f and g respectively and $Z = X + Y$ then

$$P(Z \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f(x) g(y) dy dx$$

Differentiating wrt z we get

$$f_Z(z) = \int_{-\infty}^{\infty} f(x) g(z-x) dx$$

This integral is called the **convolution** of he densities f and g.

If $T_1, \ldots, T_n$ iid Exponential(λ) then $S_n = T_1 + \cdots + T_n$ has a Gamma(n, λ) distribution. Density of S_n is

$$f_{S_n}(s) = \lambda(\lambda s)^{n-1} e^{-\lambda s}/(n-1)!$$

for $s > 0$.

Proof:

$$\begin{aligned} P(S_n > s) &= P(N(0,s] < n) \\ &= \sum_{j=0}^{n-1} (\lambda s)^j e^{-\lambda s}/j! \end{aligned}$$

Then

$$\begin{aligned} f_{S_n}(s) &= \frac{d}{ds} P(S_n \le s) \\ &= \frac{d}{ds} \{1 - P(S_n > s)\} \\ &= -\lambda \sum_{j=0} n-1 \left\{ j(\lambda s)^{j-1} - (\lambda s)^j \right\} \frac{e^{-\lambda s}}{j!} \\ &= \lambda e^{-\lambda s} \sum_{j=0}^{n-1} \left\{ \frac{(\lambda s)^j}{j!} - \frac{(\lambda s)^{j-1}}{(j-1)!} \right\} \end{aligned}$$

This telescopes to

$$f_{S_n}(s) = \lambda(\lambda s)^{n-1} e^{-\lambda s}/(n-1)!$$

Extreme Values: If $X_1, \ldots, X_n$ are independent exponential rvs with means $1/\lambda_1, \ldots, 1/\lambda_n$ then $Y = \min\{X_1, \ldots, X_n\}$ has an exponential distribution with mean

$$\frac{1}{\lambda_1 + \cdots + \lambda_n}$$

Proof:

$$\begin{aligned} P(Y > y) &= P(\forall k X_k > y) \\ &= \prod e^{-\lambda_k y} \\ &= e^{-\sum \lambda_k y} \end{aligned}$$

Memoryless Property: conditional distribution of $X - x$ given $X \geq x$ is exponential if X has an exponential distribution.

Proof:

$$\begin{aligned}
P(X - x > y | X \geq x) & \\
&= \frac{P(X > x + y, X \geq x)}{P(X > x)} \\
&= \frac{PX > x + y)}{P(X \geq x)} \\
&= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} \\
&= e^{-\lambda y}
\end{aligned}$$

Hazard Rates

The hazard rate, or instantaneous failure rate for a positive random variable T with density f and cdf F is

$$r(t) = \lim_{\delta \to 0} \frac{P(t < T \leq t + \delta | T \geq t)}{\delta}$$

This is just

$$r(t) = \frac{f(t)}{1 - F(t)}$$

For an exponential random variable with mean $1/\lambda$ this is

$$h(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

The exponential distribution has constant failure rate.

Weibull random variables have density

$$f(t|\lambda, \alpha) = \lambda(\lambda t)^{\alpha - 1} e^{-(\lambda t)^{\alpha}}$$

for $t > 0$. The corresponding survival function is

$$1 - F(t) = e^{-(\lambda t)^{\alpha}}$$

and the hazard rate is

$$r(t) = \lambda(\lambda t)^{\alpha-1}$$

which is increasing for $\alpha > 1$, decreasing for $\alpha < 1$. For $\alpha = 1$ this is the exponential distribution.

Since

$$r(t) = \frac{dF(t)/dt}{1 - F(t)} = -\frac{d\log(1 - F(t))}{dt}$$

we can integrate to find

$$1 - F(t) = \exp\{-\int_0^t r(s)ds\}$$

so that r determines F and f.