STAT 450

Final Exam

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Instructions: Open book. Notes, text, calculators, other texts permitted. A good mark requires clear explanations. The exam is out of 60.

1. Suppose that X and Y are independent and that each has density, f, given by $f(t) = 4t\exp(-2t)$ for $t > 0$ and $f(t) = 0$ for $t < 0$.

   (a) Find the joint density of $U = X/(X+Y)$ and $V = X + Y$. [4 marks]

   (b) Find the marginal density of U. [3 marks]

   (c) Find the conditional density of V given that $Y = 2$. [4 marks]

2. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed Bernoulli random variables with $P(X_1 = 1) = \theta$.

   (a) Show that $X_1(1 - X_2)$ is an unbiased estimate of $\theta(1-\theta)$. [2 marks]

   (b) Show that

   $$P(X_1(1-X_2) = 1 | \sum_{i=1}^{n} X_i = k) = k(n-k)/[n(n-1)] \, .$$

   [4 marks]

   (c) Find a UMVUE of $\theta(1-\theta)$. [4 marks]

3. Suppose $X_1, \ldots, X_n$ are independent and identically distributed with density f given by

   $$f(x) = \frac{1}{2}\theta^3 x^2 \exp(-\theta x)\mathbf{1}(x > 0) \, .$$

   Here $\theta > 0$ is a parameter.

   (a) Find the mle of θ. [5 marks]

   (b) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ. [3 marks]

   (c) Find the mle of the variance in b). [1 mark]

4. Suppose $Y_i, i = 1, \ldots, n$ are independent exponential random variables. Assume that the mean of Y_i is βx_i, where the x_i are the values of some positive covariate and $\beta > 0$ is the parameter of interest. Each part is worth 4 marks except part e) which is worth 2.

   (a) Derive the log likelihood and score functions and the Fisher information.

(b) Show that $T = \sum_1^n (Y_i/x_i)$ is sufficient.

(c) Find the density of $U_i = Y_i/x_i$ and use this to show that T has a gamma distribution with parameters n and β.

(d) Show that T/β is a pivotal quantity and show how to find a 95% confidence interval for β. For 2 bonus marks derive equations to solve to find the shortest such interval based on this pivot. Do not try to solve the equations.

(e) Show that T is complete.

(f) Find the mle of $1/\beta$. Use the answer to c) to compute the bias and mean squared error of this mle.

(g) Find the Cramér–Rao lower bound for the variance of unbiased estimates of $1/\beta$.

(h) Is there a UMP test of the hypothesis $\beta \leq 1$ against $\beta = 1$? Find it and show that it is UMP or prove no such test exists.