STAT 450
Midterm Examination I

Richard Lockhart 2 October 2002

Instructions: This is an open book exam. You may use notes, books and a calculator. The exam is out of 20. **DON'T PANIC.**

1. Suppose X, Y have joint density

$$f(x,y) = \begin{cases} kx & 0 \le x \le y \le 1 \\ ky & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is k? (3 marks)

$$\begin{aligned} k &= \int_{-\infty}^{\infty} f(x,y)dydx \\ &= k\int_0^1 \left\{ \int_0^x ydy + \int_x^1 xdy \right\} dx \\ &= k\int_0^1 \left\{ x^2/2 + x(1-x) \right\} dx \\ &= k\int_0^1 \left\{ x - x^2/2 \right\} dx \\ &= k(1/2 - 1/6) = k/3 \end{aligned}$$

   So $k = 3$.

   (b) Find the density of X. (3 marks)
   This is k times the inside integral in the first part so

$$f_X(x) = \begin{cases} 3(x - x^2/2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (c) Find the cdf of X. (1 mark)

$$\begin{aligned} F(x) &= \begin{cases} 0 & x < 0 \\ \int_0^x 3(u - u^2/2)du & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \\ &= \begin{cases} 0 & x < 0 \\ 3(x^2/2 - x^3/6) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \end{aligned}$$

(d) Find the conditional density of Y given X. (2 marks)

For $x \notin [0,1]$ the conditional density is not defined. For $x \in [0,1]$ but $y \notin [0,1]$ the conditional is 0. Otherwise:

$$f_{Y|X}(y|x) = \begin{cases} \frac{3x}{3(x-x^2/2)} & 0 \le x \le y \le 1 \\ \frac{3y}{3(x-x^2/2)} & 0 \le y \le x \le 1 \end{cases}$$

$$= \begin{cases} \frac{1}{1-x/2} & 0 \le x \le y \le 1 \\ \frac{y}{x-x^2/2} & 0 \le y \le x \le 1 \end{cases}$$

2. Suppose X_1 and X_2 are independent. Each has density f given by

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the density of $X_1 X_2$. (7 marks)

The joint density of X_1, X_2 is

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & 0 \le x_1 \le 1, 0 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Put $U = X_1X_2$ and $V = X_2$. Solve to get $X_2 = V$ and $X_1 = U/V$. The matrix of derivatives is

$$\begin{bmatrix} \partial X_1/\partial U & \partial X_1/\partial V \\ \partial X_2/\partial U & \partial X_2 \partial V \end{bmatrix} = \begin{bmatrix} 1/V & -U/V^2 \\ 0 & 1 \end{bmatrix}$$

so the Jacobian is $1/V$. We get

$$f_{UV}(u,v) = f(u/v, v)/v = \begin{cases} 4u/v & 0 \le u/v \le 1, 0 \le v \le 1 \\ 0 & \text{otherwise} \end{cases}$$

This simplifies to

$$f_{UV}(u,v) = \begin{cases} 4u/v & 0 \le u \le v \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find F_U by integrating over v. For $u < 0$ or $u > 1$ get 0. Otherwise

$$f_U(u) = \int_u^1 4u/v dv = -4u \ln u$$

Thus

$$f_U(u) = \begin{cases} -4u \ln u & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

3. Suppose Z_1 and Z_2 are independent $N(0,1)$ random variables. Let

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}.$$

HINT: There is no need to do any integrals in the following questions. You may use facts from your notes without further proof.

(a) Find a matrix A of the form

$$A = [a_1 \quad a_2]$$

so that $(Z_1 - Z_2)/\sqrt{2} = AZ$. (1 mark)

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(b) What is the distribution of $(Z_1 - Z_2)/\sqrt{2}$? (2 marks)
Since $Z \sim MVN(0, I)$, that is, Z is multivariate standard normal we have

$$(Z_1 - Z_2)/\sqrt{2} \sim N(A0, AA^T) = N(0, 1)$$

(c) What is the distribution of $(Z_1 - Z_2)^2/2$? (1 mark)
This is the square of a $N(0,1)$ variable so it is χ^2_1.