STAT 450

Solutions: Assignment 3

1. Suppose that X_1, X_2, X_3 are a sample from the Poisson(λ) distribution. Assume we see $X_1 = 1$, $X_2 = 3$ and $X_3 = 2$. Graph the likelihood and log-likelihood functions between $\lambda = 0.1$ and $\lambda = 4$.

The joint density of X_1, X_2, X_3 is

$$f(x_1, x_2, x_3; \lambda) = \frac{\lambda^{x_1}}{x_1!}e^{-\lambda}\frac{\lambda^{x_2}}{x_2!}e^{-\lambda}\frac{\lambda^{x_3}}{x_3!}e^{-\lambda}$$

which simplifies to

$$f(x_1, x_2, x_3; \lambda) = \frac{\lambda^{x_1+x_2+x_3}}{x_1!x_2!x_3!}e^{-3\lambda}$$

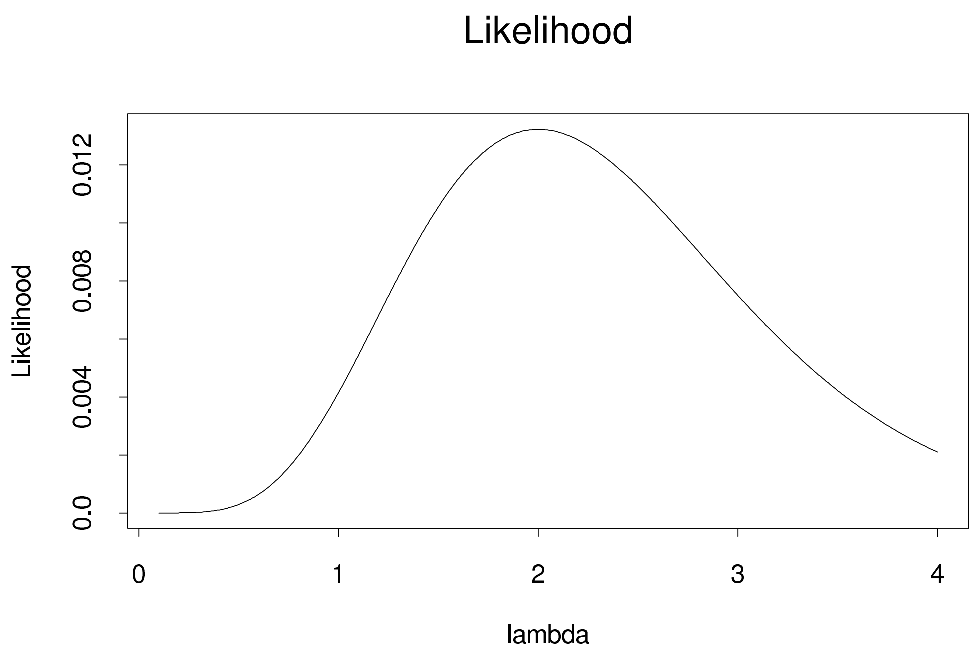
Plug in $X_1 = 1$, $X_2 = 3$ and $X_3 = 2$ to see that

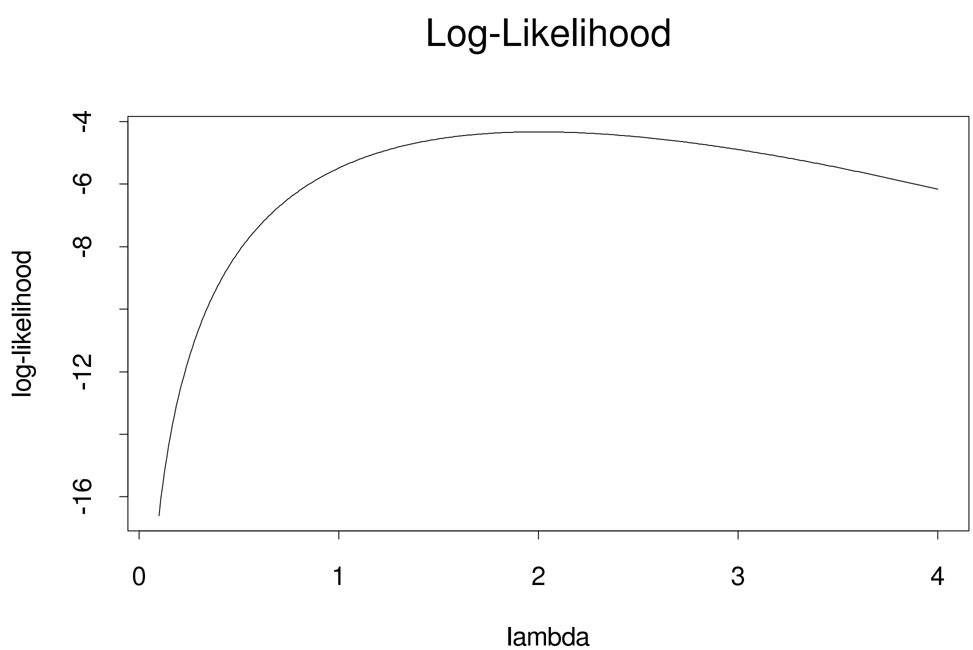
$$L(\lambda) = \frac{\lambda^6}{1!3!2!}e^{-3\lambda}$$

and

$$\ell(\lambda) = \log(L(\lambda)) = -3\lambda + 6\log(\lambda) - \log(12)$$

I plotted them using `R`:





2. Suppose that X_1, X_2, X_3 are a sample from the Uniform$[\theta, \theta+1]$ distribution. Assume we see $X_1 = 1$, $X_2 = 1.3$ and $X_3 = 0.8$.

   (a) Graph the likelihood function between $\theta = 0$ and $\theta = 2$.

   The joint density of X_1, X_2, X_3 is

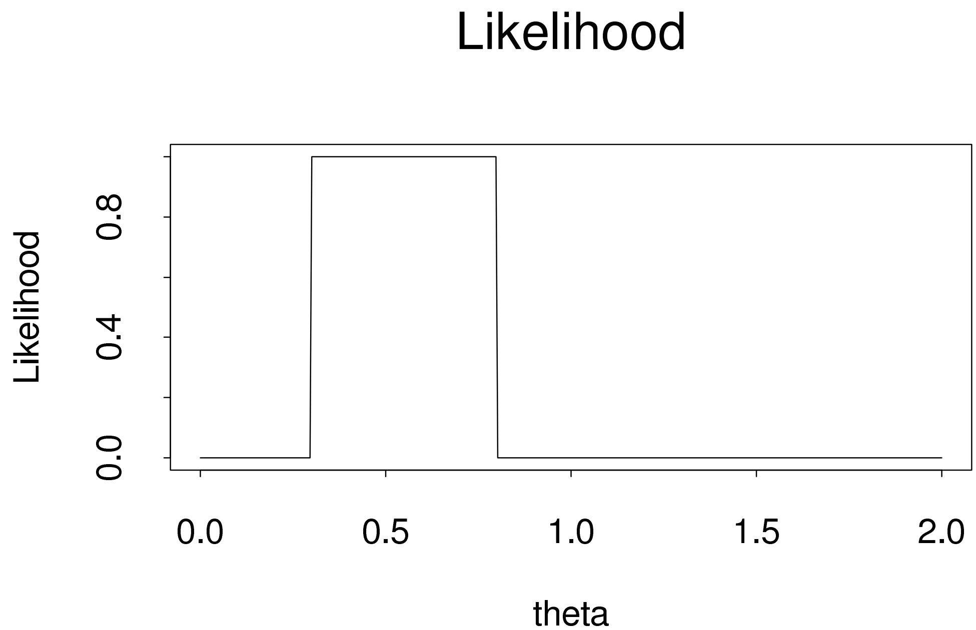
   $$f(x_1, x_2, x_3; \theta) = 1(\theta < x_1 < \theta + 1)1(\theta < x_2 < \theta + 1)1(\theta < x_3 < \theta + 1)$$

   This formula gives you 0 unless all three of the indicators give you a 1. This requires $\theta < x_i$ for $i = 1$, 2 and 3 and $x_i - 1 < \theta$ for $i = 1$, 2 and

3. The smallest x_1 is actually 0.8 and θ has to be smaller than that. The largest x_i is 1.3. Subtracting 1 you see that θ must be more that 0.3. In fact then

$$f(x_1, x_2, x_3; \theta) = 1(0.3 < \theta < 0.8)$$

Here is a plot:



(b) How would the graph change if we make $X_1 = 1.1$?

Not at all – the value of the middle X does not affect the graph. The graphs is just 1 between the largest X minus 1 and the smallest X.

(c) Is there a unique mle?

No, the likelihood is constant between 0.3 and 0.8—any such θ maximizes L.

3. For each of the following families of densities with parameter space as given, suppose you are given a sample $X_1, \ldots, X_n$. Find the likelihood, log-likelihood, the score function, the likelihood equations and the maximum likelihood estimates of the parameters. In some cases you will not be able to solve the likelihood equations completely.

(a) The Weibull family with known shape α and unknown scale β:

$$f(x; \beta) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha} 1(x > 0)$$

The parameter space is $\Theta = \{\beta > 0\}$.

Likelihood:

$$L(\beta) = \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_1^n X_i\right)^{\alpha-1} \exp\left\{-\sum X_i^\alpha/\beta^\alpha\right\} 1(\beta > 0)$$

Log-likelihood

$$\ell(\beta) = n\log(\alpha) + (\alpha-1)\sum \log(X_i) - n\alpha\log(\beta) - \sum\left(\frac{X_i}{\beta}\right)^\alpha$$

Score

$$\ell'(\beta) = -\frac{n\alpha}{\beta} + \alpha\frac{\sum X_i^\alpha}{\beta^{\alpha+1}}$$

Set this last equal to 0 to get the likelihood equation and solve to get

$$\hat\beta = \left[\frac{\sum X_i^\alpha}{n}\right]^{1/\alpha}.$$

(b) The Uniform$[0,\theta]$ family:

$$f(x,\theta) = \frac{1}{\theta}1(0 < x < \theta)$$

The parameter space is $\Theta = \{\theta > 0\}$.

The likelihood is

$$L(\theta) = \begin{cases} \theta^{-n} & X_1 < \theta, \ldots, X_n < \theta \\ 0 & \textit{otherwise.} \end{cases}$$

The log-likelihood is

$$\ell(\theta) = \begin{cases} -n\log(\theta) & X_1 < \theta, \ldots, X_n < \theta \\ -\infty & \textit{otherwise.} \end{cases}$$

The score is

$$\ell'(\theta) = \begin{cases} -n/\theta & X_1 < \theta, \ldots, X_n < \theta \\ \textit{undefined} & \textit{otherwise.} \end{cases}$$

Notice that this log-likelihood is monotone decreasing in θ but jumps from $-\infty$ up to $-n\log(\max\{X_i\})$ when θ passes the largest X. The mle is NOT found by solving the likelihood equations. It is $\hat\theta = \max\{X_i\}$.

(c) The $N(\mu,\mu^2)$ family with $\Theta = \{\mu > 0\}$.

$$L(\mu) = (2\pi)^{-n/2}|\mu|^{-n}\exp(-\sum(X_i-\mu)^2/(2\mu^2))$$

$$\ell(\mu) = -n\log(2\pi)/2 - n\log(|\mu|) - \sum(X_i-\mu)^2/(2\mu^2)$$

$$\ell'(\mu) = -\frac{n}{\mu} + \frac{\sum X_i^2}{\mu^3} - \frac{\sum X_i}{\mu^2}$$

Set this equal to 0 and multiply through by $-\mu^3$ *to get*

$$0 = n\mu^2 + \mu\sum X_i - \sum X_i^2$$

The roots are

$$\hat{\mu} = \frac{-\sum X_1 \pm \sqrt{(\sum X_i)^2 + 4n\sum X_i^2}}{2n}$$

One of these roots is negative and outside the parameter space. The MLE has the plus sign in the formula. You can check that the likelihood is 0 at 0 and at $+\infty$ *and positive in between so must have at least one local maximum over that range. This maximum must occur at a root of the likelihood equations so the mle is*

$$\hat{\mu} = \frac{-\bar{X} + \sqrt{\bar{X}^2 + 4\overline{X^2}}}{2}$$

(d) The Uniform$[\theta, \theta+1]$ family.

Go back to question 1 to see that

$$L(\theta) = 1(\max\{X_i\} - 1 < \theta < \min\{X_i\})$$

$$\ell(\theta) = \begin{cases} 0 & \max\{X_i\} - 1 < \theta < \min\{X_i\} \\ -\infty & \textit{otherwise} \end{cases}$$

$$\ell'(\theta) = \begin{cases} 0 & \max\{X_i\} - 1 < \theta < \min\{X_i\} \\ \textit{undefined} & \textit{otherwise} \end{cases}$$

The log-likelihood is constant for all θ *in the range* $(\max\{X_i\}-1, \min\{X_i\})$ *so the mle is not unique.*

4. Suppose that $Y_1, \ldots, Y_n$ are independent random variables and that $x_1, \ldots, x_n$ are the corresponding values of some covariate. Suppose that the density of Y_i is

$$f(y_i) = \exp\left(-y_i\exp(-\alpha - \beta x_i) - \alpha - \beta x_i\right) 1(y_i > 0)$$

where α, and β are unknown parameters. Find the log-likelihood, the score function and the likelihood equations.

$$\ell(\alpha, \beta) = -\sum [Y_i \exp(-\alpha - \beta x_i) + \alpha + \beta x_i]$$

The score function has two components:

$$\frac{\partial \ell}{\partial \alpha} = \sum Y_i \exp(-\alpha - \beta x_i) - n$$

$$\frac{\partial \ell}{\partial \beta} = \sum [Y_i x_i \exp(-\alpha - \beta x_i) - x_i]$$

Setting these two equal to 0 gives the likelihood equations.

5. For each of the doses $d_1, \ldots, d_p$ a number of animals $n_1, \ldots, n_p$ are treated with the corresponding dose of some drug. The number dying at dose d is Binomial with parameter $h(d)$. A common model for $h(d)$ is $\log(h/(1-h)) = \alpha + \beta d$. Find the likelihood equations for estimating α and β.

Let X_i be the number dying at dose d_i. Then

$$L(\alpha, \beta) = \prod_{i=1}^{n} \binom{n}{X_i} h(d_i)^{X_i} (1 - h(d_i))^{n_i - X_i}$$

The log likelihood is

$$\begin{aligned} \ell(\alpha, \beta) &= \sum \left\{ \log \left[\binom{n}{X_i} \right] + n_i \log(1 - h(d_i)) + X_i \log(h(d_i)/(1 - h(d_i))) \right\} \\ &= \sum \left\{ \log \left[\binom{n}{X_i} \right] + n_i \log(1 - h(d_i)) + X_i(\alpha + \beta d_i) \right\} \end{aligned}$$

To get the likelihood equations you take derivatives:

$$\frac{\partial \ell}{\partial \alpha} = \sum \left[X_i - \frac{n \frac{\partial}{\partial \alpha} h(d_i)}{1 - h(d_i)} \right]$$

and

$$\frac{\partial \ell}{\partial \beta} = \sum \left[X_i d_i - \frac{n \frac{\partial}{\partial \beta} h(d_i)}{1 - h(d_i)} \right]$$

and set these equal to 0. To simplify these differentiate the relation

$$\log(h/(1-h)) \log(h) - \log(1-h) = \alpha + \beta d.$$

to get

$$\frac{\frac{\partial h}{\partial \alpha}}{h} + \frac{\frac{\partial h}{\partial \alpha}}{1-h} = 1$$

and

$$\frac{\frac{\partial h}{\partial \beta}}{h} + \frac{\frac{\partial h}{\partial \beta}}{1-h} = d$$

Then rearrange to get

$$\frac{\partial h}{\partial \alpha} = h(d)(1 - h(d))$$

and

$$\frac{\partial h}{\partial \beta} = dh(d)(1 - h(d))$$

The likelihood equations are

$$\frac{\partial \ell}{\partial \alpha} = \sum [X_i - n_i h(d_i)] = 0$$

and

$$\frac{\partial \ell}{\partial \beta} = \sum [X_i d_i - n_i d_i h(d_i)] = 0\,.$$