STAT 801 Lecture 23

Reading for Today's Lecture:
Goals of Today's Lecture:

- Introduce confidence sets.
- Describe relation of tests to confidence intervals.
- Describe the use of pivots to make confidence sets.
- Introduce decision theory.

Today's notes

Confidence Sets

A level β confidence set for a parameter $\phi(\theta)$ is a random subset C, of the set of possible values of ϕ such that for each θ we have

$$P_\theta(\phi(\theta) \in C) \geq \beta$$

Confidence sets are very closely connected with hypothesis tests:

Suppose C is a level $\beta = 1 - \alpha$ confidence set for ϕ. To test $\phi = \phi_0$ we consider the test which rejects if $\phi \notin C$. This test has level α. Conversely, suppose that for each ϕ_0 we have available a level α test of $\phi = \phi_0$ who rejection region is say R_{ϕ_0}. Then if we define $C = \{\phi_0 : \phi = \phi_0 \text{ is not rejected}\}$ we get a level $1 - \alpha$ confidence for ϕ. The usual t test gives rise in this way to the usual t confidence intervals

$$\bar{X} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}$$

which you know well.

Confidence sets from Pivots

Definition: A **pivot** (or pivotal quantity) is a function $g(\theta, X)$ whose distribution is the same for all θ. (As usual the θ in the pivot is the same θ as the one being used to calculate the distribution of $g(\theta, X)$.

Pivots can be used to generate confidence sets as follows. Pick a set A in the space of possible values for g. Let $\beta = P_\theta(g(\theta, X) \in A)$; since g is pivotal β is the same for all θ. Now given a data set X solve the relation

$$g(\theta, X) \in A$$

to get

$$\theta \in C(X, A)\,.$$

Example: The quantity

$$(n-1)s^2/\sigma^2$$

is a pivot in the $N(\mu, \sigma^2)$ model. It has a χ^2_{n-1} distribution. Given $\beta = 1-\alpha$ consider the two points $\chi^2_{n-1,1-\alpha/2}$ and $\chi^2_{n-1,\alpha/2}$. Then

$$P(\chi^2_{n-1,1-\alpha/2} \leq (n-1)s^2/\sigma^2 \leq \chi^2_{n-1,\alpha/2}) = \beta$$

for all μ, σ. We can solve this relation to get

$$P(\frac{(n-1)^{1/2}s}{\chi_{n-1,\alpha/2}} \leq \sigma \leq \frac{(n-1)^{1/2}s}{\chi_{n-1,1-\alpha/2}}) = \beta$$

so that the interval from $(n-1)^{1/2}s/\chi_{n-1,\alpha/2}$ to $(n-1)^{1/2}s/\chi_{n-1,1-\alpha/2}$ is a level $1-\alpha$ confidence interval.

In the same model we also have

$$P(\chi^2_{n-1,1-\alpha} \leq (n-1)s^2/\sigma^2) = \beta$$

which can be solved to get

$$P(\sigma \leq \frac{(n-1)^{1/2}s}{\chi_{n-1,1-\alpha/2}}) = \beta$$

This gives a level $1-\alpha$ interval $(0, (n-1)^{1/2}s/\chi_{n-1,1-\alpha})$. The right hand end of this interval is usually called a confidence upper bound.

In general the interval from $(n-1)^{1/2}s/\chi_{n-1,\alpha_1}$ to $(n-1)^{1/2}s/\chi_{n-1,1-\alpha_2}$ has level $\beta = 1-\alpha_1-\alpha_2$. For a fixed value of β we can minimize the length of the resulting interval numerically. This sort of optimization is rarely used. See your homework for an example of the method.