STAT 804

Lecture 22 Notes

Properties of the Periodogram

The discrete Fourier transform

$$\hat{X}(\omega) = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} X_t \exp(2\pi\omega t i)$$

is periodic with period 1 because all the exponentials have period 1. Moreover,

$$\hat{X}(1-\omega) = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} X_t \exp(-2\pi\omega t i)\exp(2\pi t i) = \overline{\hat{X}(\omega)}$$

so that the periodogram satisfies

$$|\hat{X}(1-\omega)|^2 = |\hat{X}(\omega)|^2$$

Thus the periodogram is symmetric around $\omega = 1/2$ which is called the *Nyquist* or *folding* frequency. (The value is always 1/2 in cycles per point but usually it would be converted to cycles per time unit like year or day or whatever.)

Similarly the power spectral density f_X given by

$$f_X(\omega) = \sum_{-\infty}^{\infty} C_X(h) \exp(2\pi h\omega i)$$

is periodic with period 1 and satisfies

$$f_X(-\omega) = f_X(\omega)$$

which is equivalent to

$$f_X(1-\omega) = f_X(\omega)\,.$$

Spectra of Some Basic Processes

Here I compute the spectra of a few basic processes directly and then indirectly by a more powerful technique.

Direct from the definition

White Noise: Since $C_\epsilon(k) = 0$ for all non-zero k we have

$$f_\epsilon(\omega) \equiv C_\epsilon(0) = \sigma_\epsilon^2\,.$$

MA(1): For $X_t = \epsilon_t - b\epsilon_{t-1}$ we have $C_X(0) = \sigma^2(1+b^2)$ and $C_X(\pm 1) = -b\sigma^2$ so that

$$\begin{aligned} f_X(\omega) &= \sigma^2(1+b^2) - b\sigma^2(\exp(2\pi\omega i) + \exp(-2\pi\omega i)) \\ &= \sigma^2(1+b^2) - 2b\sigma^2\cos(2\pi\omega) \end{aligned}$$

AR(1): We have $C_X(k) = \rho^{|k|}C_X(0)$ and

$$\begin{aligned} f_X(\omega) &= C_X(0)(1 + \sum_{k>0} \rho^k(\exp(2\pi\omega k i) + (\exp(-2\pi\omega k i)) \\ &= C_X(0)(1 + \rho(\frac{e^{2\pi\omega i}}{1-\rho\exp(2\pi\omega i)} + \frac{e^{-2\pi\omega i}}{1-\rho\exp(-2\pi\omega i)})) \\ &= C_X(0)(1 + \rho\frac{\exp(2\pi\omega i) - \rho + \exp(-2\pi\omega i) - \rho}{(1-\rho\exp(2\pi\omega i))(1-\rho\exp(-2\pi\omega i))} \\ &= C_X(0)(1 + \rho\frac{2(\cos(2\pi\omega) - \rho))}{1+\rho^2 - 2\rho\cos(2\pi\omega)} \\ &= C_X(0)\frac{1-\rho^2}{1+\rho^2-2\rho\cos(2\pi\omega)} \\ &= \frac{\sigma_\epsilon^2}{1+\rho^2-2\rho\cos(2\pi\omega)} \end{aligned}$$

Using filters

Any mean 0 ARMA(p,q) process can be rewritten in MA form as

$$X_t = \sum_{s=0}^{\infty} a_s \epsilon_{t-s}$$

and then the covariance of X is

$$C_X(h) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_r a_s Cov(\epsilon_{t+h-r}, \epsilon_{t-s})$$

Although the covariance simplifies for white noise, let us simply write $C(t+h-r-(t-s)) = C(h+s-r)$ for the covariance in this double sum so that the calculation will apply to any stationary ϵ. Then plug this double sum into the definition of f_X to get

$$f_X(\omega) = \sum_{h=-\infty}^{\infty} \exp(2\pi\omega h i) \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_r a_s C(h+s-r)$$

Now write the h in the complex exponential in the form $(h+s-r)+r-s$ and bring the sum over h to the inside to get

$$f_X(\omega) = \sum_{r=0}^{\infty} a_r e^{2\pi\omega r i} \sum_{s=0}^{\infty} a_s e^{-2\pi\omega s i} \sum_{h=-\infty}^{\infty} \exp(2\pi\omega(h+s-r)i) C(h+s-r)$$

Finally make the substitution $k = h + s - r$ in the inside sum and define

$$A(\omega) = \sum_{r=0}^{\infty} a_r e^{2\pi\omega r i}$$

to see that

$$f_X(\omega) = A(\omega)\overline{A(\omega)} f_\epsilon(\omega)$$

or

$$f_X(\omega) = |A(\omega)|^2 f_\epsilon(\omega)\,.$$

The function A (or $\bar{A}$) is called the *frequency response function* and $|A|^2$ the *power transfer function.* The jargon *gain* is sometimes used for $|A|$.

The Spectrum of an ARMA(p, q)

An ARMA(p, q) process X satisfies

$$\sum_{s=0}^{p} a_s X_{t-s} = \sum_{r=0}^{q} b_r \epsilon_{t-r}$$

so that if Y is the process $\sum_{s=0}^{p} a_s X_{t-s}$ then

$$f_Y(\omega) = |A(\omega)|^2 f_X(\omega)$$

where $A(\omega) = \sum_{s=0}^{p} a_s \exp(2\pi\omega s i)$. At the same time $Y_t = \sum_{r=0}^{q} b_r \epsilon_{t-r}$ so

$$f_Y(\omega) = |B(\omega)|^2 f_\epsilon(\omega)$$

where $B(\omega) = \sum_{s=0}^{q} b_s \exp(2\pi\omega s i)$. Hence

$$f_X(\omega) = \frac{|B(\omega)|^2}{|A(\omega)|^2}\,.$$

For example, in the ARMA(1,1) case $X_t - aX_{t-1} = \epsilon_t - b\epsilon_{t-1}$ we find (referring to our MA(1) calculation above that $|B(\omega)|^2 = 1 + b^2 - 2b\cos(2\pi\omega)$ and $|A(\omega)|^2 = 1 + a^2 - 2a\cos(2\pi\omega)$ so that

$$f_X(\omega) = \frac{1 + b^2 - 2b\cos(2\pi\omega)}{1 + a^2 - 2a\cos(2\pi\omega}\,.$$