STAT 804: Notes on Lecture 4

More than 1 process

Definition: Two processes X and Y are jointly (strictly) stationary if

$$\mathcal{L}(X_t, \ldots, X_{t+h}, Y_t, \ldots, Y_{t+h}) = \mathcal{L}(X_0, \ldots, X_h, Y_0, \ldots, Y_h)$$

for all t and h. They are jointly second order stationary if each is second order stationary and also

$$C_{XY}(h) \equiv \mathrm{Cov}(X_t, Y_{t+h}) = \mathrm{Cov}(X_0, Y_h)$$

for all t and h. Notice that negative values of h give, in general, different covariances than positive values of h.
Definition: If X is stationary then we call $C_X(h) = \mathrm{Cov}(X_0, X_h)$ the **autocovariance** function of X.
Definition: If X and Y are jointly stationary then we call $C_{XY}(h) = \mathrm{Cov}(X_0, Y_h)$ the **cross-covariance** function.

Notice that $C_X(-h) = C_X(h)$ and $C_{XY}(h) = C_{YX}(-h)$ for all h and similarly for correlation
Definition: The **autocorrelation** function of X is

$$\rho_X(h) = C_X(h)/C_X(0) \equiv \mathrm{Corr}(X_0, X_h)\,.$$

the **cross-correlation** function of X and Y is

$$\rho_{XY}(h) = \mathrm{Corr}(X_0, Y_h) = C_{XY}(h)/\sqrt{C_X(0)C_Y(0)}\,.$$

Fact: If X and Y are jointly stationary then $aX + bY$ is stationary for any constants a and b.

Model Identification

The goal of this section is to develop tools to permit us to choose a model for a given series X. We will be attempting to fit an $ARMA(p, q)$ and our first step is to learn how to choose p and q. We will try to get small values of these orders and our efforts are focused on the cases with either p or q equal to 0. We use the autocorrelation or autocovariance function to do model identification.

Some Theoretical Autocovariances

1. Moving Averages. Since addition of a constant never affects a covariance we take the mean equal to 0 and look at

$$X_t = \epsilon_t - \sum_1^p b_j \epsilon_{t-j}$$

Using $b_0 = 1$ we find

$$\begin{aligned} C_X(h) &= \mathrm{Cov}(X_t, X_{t+h}) \\ &= \mathrm{Cov}(\sum_{j=0}^p b_j \epsilon_{t-j}, \sum_{k=0}^p b_k \epsilon_{t+h-k}) \\ &= \sum_{j=0}^p \sum_{k=0}^p b_j b_k \mathrm{Cov}(\epsilon_{t-j}, \epsilon_{t+h-k}) \end{aligned}$$

Each covariance is 0 unless $t - j = t + h - k$ or $k = j + h$. This gives

$$\begin{aligned} C_X(h) &= \sigma^2 \sum_{j=0}^p \sum_{k=0}^p b_j b_k 1(k = j + h) \\ &= \sigma^2 \sum_{j=0}^p b_j b_{j+h} 1(0 \le j + h \le p) \\ &= \sigma^2 \sum_{j=0}^{p-h} b_j b_{j+h} \end{aligned}$$

Notice that if $h > p$ (or $h < -p$) then we get $C_X(h) = 0$.

Jargon: We call h the lag and say that for an $MA(p)$ process the autocovariance function is 0 at lags larger than p.

To identify an $MA(p)$ look at a graph of an estimate $\hat{C}(h)$ and look for a lag where it suddenly decreases to (nearly) 0.

2. Autoregressive Processes. Again we take $\mu = 0$. Consider first $p = 1$ so that $X_t = \rho X_{t-1} + \epsilon_t$. Then

$$\begin{aligned} C_X(h) &= \mathrm{Cov}(X_t, X_{t+h}) \\ &= \mathrm{Cov}(X_t, \rho X_{t+h-1} + \epsilon_{t+h}) \\ &= \rho \mathrm{Cov}(X_t, X_{t+h-1}) + \mathrm{Cov}(X_t, \epsilon_{t+h}) \end{aligned}$$

For $h > 0$ the term $\mathrm{Cov}(X_t, \epsilon_{t+h}) = 0$. This gives

$$\begin{aligned} C_X(h) &= \rho C_X(h-1) \\ &= \rho^2 C_X(h-2) \\ &\ \ \vdots \\ &= \rho^h C_X(0) \end{aligned}$$

This gives

$$\rho_X(h) = \rho_X(1)^h = \rho^h$$

You should also recall that $C_X(0) = \sigma^2/(1-\rho^2)$.

Notice that $R_X(h)$ decreases geometrically to 0 but is never actually 0.

Remark: If ρ is small so that ρ^2 is very small then an $AR(1)$ process is approximately the same as an $MA(1)$ process: we nearly have $X_t = \epsilon_t + \rho\epsilon_{t-1}$.