STAT 830
Non-parametric Inference Basics

Richard Lockhart

Simon Fraser University

STAT 830 — Fall 2013

The Empirical Distribution Function – EDF pp 97-99

- Suppose we have sample $X_1, \ldots, X_n$ of iid real valued rvs.
- The empirical distribution function is

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1(X_i \le x)$$

- This is a cdf and is an estimate of F, the cdf of the Xs.
- People also speak of the empirical distribution:

$$\hat{P}(A) = \frac{1}{n} \sum_{i=1}^{n} 1(X_i \in A)$$

- This is the probability distribution corresponding to $\hat{F}_n$.
- Now we consider the qualities of $\hat{F}_n$ as an estimate, the standard error of the estimate, the estimated standard error, confidence intervals, simultaneous confidence intervals and so on.

Bias, variance and mean squared error

- Judge estimates in many ways; focus is distribution of error $\hat{\theta} - \theta$.
- Distribution computed when θ is *true* value of parameter.
- For our non-parametric iid sampling model we are interested in

$$\hat{F}_(x) - F(x)$$

  when F is the true distribution function of the Xs.
- Simplest summary of size of a variable is root mean squared error:

$$RMSE = \sqrt{\mathrm{E}_\theta\left[(\hat{\theta} - \theta)^2\right]}$$

- Subscript θ on E is important – specifies true value of θ and matches θ in the error!

MSE decomposition & variance-bias trade-off

- The MSE of any estimate is

$$\begin{aligned} MSE &= \mathrm{E}_\theta\left[(\hat\theta-\theta)^2\right] \\ &= \mathrm{E}_\theta\left[(\hat\theta-\mathrm{E}_\theta(\hat\theta)+\mathrm{E}_\theta(\hat\theta)-\theta)^2\right] \\ &= \mathrm{E}_\theta\left[(\hat\theta-\mathrm{E}_\theta(\hat\theta))^2\right]+\left\{\mathrm{E}_\theta(\hat\theta)-\theta\right\}^2 \end{aligned}$$

- In making this calculation there was a cross product term which is 0.
- The two terms each have names: the first is the variance of $\hat\theta$ while the second is the square of the bias.
- **Definition**: The **bias** of an estimator $\hat\theta$ is

$$\mathrm{bias}_{\hat\theta}(\theta)=\mathrm{E}_\theta(\hat\theta)-\theta$$

- So our decomposition is

$$MSE = \mathrm{Variance} + (\mathrm{bias})^2.$$

- In practice often find a trade-off. Trying to make the variance small increases the bias.

Applied to the EDF

- The EDF is an *unbiased* estimate of F. That is:

$$\begin{aligned}\mathrm{E}[\hat{F}_n(x)] &= \frac{1}{n}\sum_{i1=}^{n}\mathrm{E}[1(X_i \le x)] \\ &= \frac{1}{n}\sum_{i=1}^{n} F(x) = F(x)\end{aligned}$$

  so the bias is 0.
- The mean squared error is then

$$\mathrm{MSE} = \mathrm{Var}(\hat{F}_n(x)) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}[1(X_i \le x)] = \frac{1}{n}F(x)[1-F(x)].$$

- This is very much the most common situation: the MSE is proportional to $1/n$ in large samples.
- So the RMSE is proportional to $1/\sqrt{n}$.
- RMSE is measured in same units as $\hat{\theta}$ so is scientifically right.

Biased estimates

- Many estimates exactly or approximately averages or ftns of averages.
- So, for example,

$$\bar{X} = \frac{1}{n} X_i \quad \text{and} \quad \bar{X^2} = \frac{1}{n} X_i^2$$

are unbiased estimates of $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$.

- We might combine these to get a natural estimate of σ^2:

$$\hat{\sigma}^2 = \bar{X^2} - \bar{X}^2$$

- This estimate is biased:

$$\mathrm{E}\left[(\bar{X})^2\right] = \mathrm{Var}(\bar{X}) + \left[\mathrm{E}(\bar{X})\right]^2 = \sigma^2/n + \mu^2.$$

So the bias of $\hat{\sigma}^2$ is

$$\mathrm{E}\left[\bar{X^2}\right] - \mathrm{E}\left[(\bar{X})^2\right] - \sigma^2 = \mu_2' - \mu^2 - \sigma^2/n - \sigma^2 = -\sigma^2/n.$$

Relative sizes of bias and variance

- In this case and many others the bias is proportional to $1/n$.
- The variance is proportional to $1/n$.
- The squared bias is proportional to $1/n^2$.
- So in large samples the variance is more important!
- The biased estimate $\hat{\sigma}^2$ is traditionally changed to the usual sample variance $s^2 = n\hat{\sigma}^2/(n-1)$ to remove the bias.
- WARNING: the MSE of s^2 is larger than that of $\hat{\sigma}^2$.

Standard Errors and Interval Estimation

- In any case point estimation is a silly exercise.
- Assessment of likely size of error of estimate is essential.
- A confidence interval is one way to provide that assessment.
- The most common kind is approximate:

$$\text{estimate} \pm 2 \text{ estimated } \textbf{standard error}$$

- This is an interval of values $L(X) <$ parameter $< U(X)$ where U and L are random.
- Justification for the two se interval above?
- Notation $\hat\phi$ is the estimate of ϕ. $\hat\sigma_{\hat\phi}$ is the estimated standard error.
- Use central limit theorem, delta method, Slutsky's theorem to prove

$$\lim_{n\to\infty} P_F\left(\frac{\hat\phi-\phi}{\hat\sigma_{\hat\phi}} \le x\right) = \Phi(x)$$

Pointwise limits for $F(x)$

- Define, as usual z_α by $\Phi(z_\alpha) = 1 - \alpha$ and approximate

$$P_F\left(-z_{\alpha/2} \le \frac{\hat\phi - \phi}{\hat\sigma_{\hat\phi}} \le z_{\alpha/2}\right) \approx 1 - \alpha.$$

- Solve inequalities to get usual interval.
- Now we apply this to $\phi = F(x)$ for one fixed x.
- Our estimate is $\hat\phi \equiv \hat F_n(x)$.
- The random variable $n\hat\phi$ has a Binomial distribution.
- So $\mathrm{Var}(\hat F_n(x)) = F(x)(1 - F(x))/n$. The standard error is

$$\sigma_{\hat\phi} \equiv \sigma_{\hat F_n(x)} \equiv \mathrm{SE} \equiv \frac{\sqrt{F(x)[1 - F(x)]}}{\sqrt{n}}.$$

- According to the central limit theorem

$$\frac{\hat F_n(x) - F(x)}{\sigma_{\hat F_n(x)}} \xrightarrow{d} N(0,1)$$

- See homework to turn this into a confidence interval.

Plug-in estimates

- Now to estimate the standard error.
- It is easier to solve the inequality

$$\left| \frac{\hat{F}_n(x) - F(x)}{\mathrm{SE}} \right| \le z_{\alpha/2}$$

  if the term SE does not contain the unknown quantity $F(x)$.
- This is why we use an estimated standard error.
- In our example we will estimate $\sqrt{F(x)[1-F(x)]/n}$ by replacing $F(x)$ by $\hat{F}_n(x)$:

$$\hat{\sigma}_{F_n(x)} = \sqrt{\frac{\hat{F}_n(x)[1-\hat{F}_n(x)}{n}}.$$

- This is an example of a general strategy: *plug-in*.
- Start with estimator, confidence interval or test whose formula depends on other parameter; plug-in estimate of that other parameter.
- Sometimes the method changes the behaviour of our procedure and sometimes, at least in large samples, it doesn't.

Pointwise versus Simultaneous Confidence Limits

- In our example Slutsky's theorem shows
$$\frac{\hat{F}_n(x) - F(x)}{\hat{\sigma}_{F_n(x)}} \xrightarrow{d} N(0,1).$$
- So there was no change in the limit *law* (alternative jargon for distribution).
- We now have two pointwise 95% confidence intervals:
$$\hat{F}_n(x) \pm z_{0.025}\sqrt{\hat{F}_n(x)[1-\hat{F}_n(x)]/n}$$
or
$$\{F(x) : \left|\frac{\sqrt{n}(\hat{F}_n(x) - F(x))}{\sqrt{F(x)[1-F(x)]}}\right| \le z_{0.025}\}$$
- When we use these intervals they depend on x.
- And we usually look at a plot of the results against x.
- If we pick out an x for which the confidence interval is surprising to us we may well be picking one of the x values for which the confidence interval misses its target.

Simultaneous intervals

- So we really want

$$P_F(L(X,x) \le F(x) \le U(X,x) \text{ for all } x) \ge 1-\alpha.$$

- In that case the confidence intervals are called *simultaneous*.
- Two possible methods: one exact, but conservative, one approximate, less conservative.
- Dvoretsky-Kiefer-Wolfowitz inequality:

$$P_F(\exists x : |\hat{F}_n(x) - F(x)| > \sqrt{\frac{-\log(\alpha/2)}{2n}}) \le \alpha$$

- Limit theory:

$$P_F(\exists x : |\sqrt{n}|\hat{F}_n(x) - F(x)| > y) \to P(\exists x : |B_0(x)| > y)$$

where B_0 is a *Brownian Bridge* (special Gaussian process).

Statistical Functionals

- Not all parameters are created equal.
- In the Weibull model density

$$f(x;\alpha,\beta) = \frac{1}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\{-(x/\beta)^\alpha\}1(x>0).$$

  there are two parameters: shape α and scale β.
- These parameters have no meaning in other densities.
- But every distribution has a median and other quantiles:

$$p^{\text{th}}\text{-quantile} = \inf\{x : F(x) \ge p\}.$$

- If r is bounded ftn then every distribution has value for parameter

$$\phi \equiv \mathrm{E}_F(r(X)) \equiv \int r(x)dF(x).$$

- Most distributions have a mean, variance and so on.
- A ftn from set of all cdfs to real line is called a *statistical functional*.
- Example: $\mathrm{E}_F(X^2) - [\mathrm{E}_F(X)]^2$.

Statistical functionals

- The statistical functional
$$T(F) = \int r(x) dF(x)$$
is linear.
- The sample variance is not a linear functional.
- Statistical functionals are often estimated using plug-in estimates so
$$T(\hat{F}) = \int r(x) d\hat{F}_n(x) = \frac{1}{n} \sum_1^n r(X_i).$$
- This estimate is unbiased and has variance
$$\sigma^2_{T(\hat{F})} = n^{-1} \left[\int r^2(x) dF(x) - \left\{ \int r(x) dF(x) \right\}^2 \right].$$
- This can in turn be estimated using a plug-in estimate:
$$\hat{\sigma}^2_{T(\hat{F})} = n^{-1} \left[\int r^2(x) d\hat{F}_n(x) - \left\{ \int r(x) d\hat{F}_n(x) \right\}^2 \right].$$

Plug-in estimates of functionals; bootstrap standard errors

- When $r(x) = x$ we have $T(F) = \mu_F$ (the mean)
- The standard error is $\sigma/\sqrt{n}$.
- Plug-in estimate of SE replaces σ with sample SD (with n not $n-1$ as the divisor).
- Now consider a general functional $T(F)$.
- The plug-in estimate of this is $T(\hat{F}_n)$.
- The plug-in estimate of the standard error of this estimate is

$$\sqrt{\mathrm{Var}_{\hat{F}_n}(T(\hat{F}_n))}.$$

which is hard to read and seems hard to calculate in general.

- The solution is to simulate, particularly to estimate the standard error.

Basic Monte Carlo

- To compute a probability or expected value can simulate.
- **Example**: To compute $P(|X| > 2)$ use software to generate some number, say M, of replicates: $X_1^*, \ldots, X_M^*$ all having same distribution as X.
- Estimate desired probability using sample fraction.
- R code: `x=rnorm(1000000) ; y =rep(0,1000000); y[abs(x) >2] =1 ; sum(y)`
- Produced 45348 when I tried it. Gives $\hat{p} = 0.045348$.
- Correct answer is 0.04550026.
- Using a million samples gave 2 correct digits, error of 2 in third digit.
- Using $M = 10000$ is more common. I got $\hat{p} = 0.0484$.
- SE of $\hat{p}$ is $\sqrt{p(1-p)}/100 = 0.0021$. So error of up to 4 in second significant digit is likely.

The bootstrap

- In bootstrapping X is replaced by the whole data set.
- Generate new data sets (X^*) from distribution F of X.
- Don't know F so use $\hat{F}_n$.
- **Example**: Interested in distribution of t pivot:

$$t = \frac{\sqrt{n}(\bar{X} - \mu)}{s}.$$

- Have data $X_1, \ldots, X_n$. Don't know μ or cdf of Xs.
- Replace these by quantities computed from $\hat{F}_n$.
- Call $\mu^* = \int x d\hat{F}_n(x) = \bar{X}$.
- Draw $X^*_{1,1}, \ldots, X^*_{1,n}$ an iid sample from the cdf $\hat{F}$.
- Repeat M times computing t from * values each time.

Bootstrapping the t pivot

- Here is R code:

```
x=runif(5)
mustar = mean(x)
tv=rep(0,M)
tstarv=rep(0,M)
for( i in 1:M){
   xn=runif(5)
   tv[i]=sqrt(5)*mean(xn-0.5)/sqrt(var(xn))
   xstar=sample(x,5,replace=TRUE)
   tstarv[i]=sqrt(5)*mean(xstar-mustar)/sqrt(var(xstar))
}
```

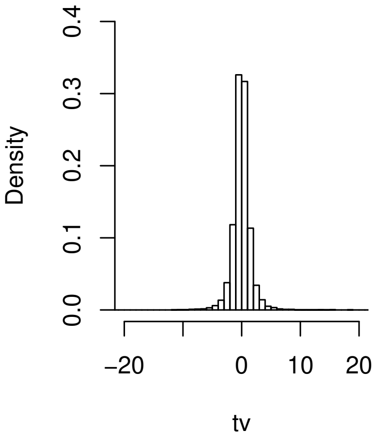
Bootstrapping a pivot continued

- Loop does two simulations.
- in `xn` and `tv` we do *parametric bootstrapping*: simulate t-pivot from parametric model.
- `xstar` is bootstrap sample from population `x`.
- `tstarv` is t-pivot computed from `xstar`.
- Original data set is

$$(0.7432447, 0.8355277, 0.8502119, 0.3499080, 0.8229354)$$

- So `mustar` $=0.7203655$
- Side by side histograms of `tv` and `tstarv` on next slide.

Bootstrap distribution histograms



Using the bootstrap distribution

- Confidence intervals: based on t-statistic: $T = \sqrt{n}(\bar{X} - \mu)/s$.
- Use the bootstrap distribution to estimate $P(|T| > t)$.
- Adjust t to make this 0.05. Call result c.
- Solve $|T| < c$ to get interval

$$\bar{X} \pm cs/\sqrt{n}.$$

- Get $c = 22.04$, $\bar{x} = 0.720$, $s = 0.211$; interval is -1.36 to 2.802.
- Pretty lousy interval. Is this because it is a bad idea?
- Repeat but simulate $\bar{X}^* - \mu^*$.
- Learn

$$P(\bar{X}^* - \mu^* < -0.192) = 0.025 = P(\bar{X}^* - \mu^* > 0.119)$$

- Solve inequalities to get (much better) interval

$$0.720 - 0.119 < \mu < 0.720 + 0.192$$

- Of course the interval missed the true value!

Monte Carlo Study

- So how well do these methods work?
- Theoretical analysis: let C_n be resulting interval.
- Assume number of bootstrap reps is so large that we can ignore simulation error.
- Compute
$$\lim_{n\to\infty} P_F(\mu(F) \in C_n)$$
- Method is *asymptotically valid* (or calibrated or accurate) if this limit is $1-\alpha$.
- Simulation analysis: generate many data sets of size 5 from Uniform.
- Then bootstrap each data set, compute C_n.
- Count up number of simulated uniform data sets with $0.5 \in C_n$ to get coverage probability.
- Repeat with (many) other distributions.

R code

```
tstarint = function(x,M=10000){
n = length(x)
must=mean(x)
se=sqrt(var(x)/n)
xn=matrix(sample(x,n*M,replace=T),nrow=M)
one = rep(1,n)/n
dev= xn%*%one - must
tst=dev/sqrt(diag(var(t(xn)))/n)
c1=quantile(dev,c(0.025,0.975))
c2=quantile(abs(tst),0.95)
c(must-c1[2],must-c1[1], must -c2*se,must+c2*se)
}
```

R code

```
lims=matrix(0,1000,4)
count=lims
for(i in 1:1000){
x=runif(5)
lims[i,]=tstarint(x)
}
count[,1][lims[,1]<0.5]=1
count[,2][lims[,2]>0.5]=1
count[,3][lims[,3]<0.5]=1
count[,4][lims[,4]>0.5]=1
sum(count[,1]*count[,2])
sum(count[,3]*count[,4])
```

Results

- 804 out of 1000 intervals based on $\bar{X} - \mu$ cover the true value of 0.5.
- 972 out of 1000 intervals based on t statistics cover true value.
- This is the uniform distribution.
- Try another distribution. For exponential I get 909, 948.
- Try another sample size. For uniform $n = 25$ I got 921, 941.