The Forty-Sixth Annual William Lowell Putnam Competition
Saturday, December 7, 1985

Done all

A–1 Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

Solution: *For each integer i from 1 to 10 you must put i in one of 6 sets: $A_1 \cap A_2^c \cap A_3^c$ or $A_1 \cap A_2^c \cap A_3$ or $A_1 \cap A_2 \cap A_3^c$ or $A_1^c \cap A_2 \cap A_3$ or $A_1^c \cap A_2^c \cap A_3$ or $A_1^c \cap A_2 \cap A_3^c$. Thus there are 6 choices for each of the 10 values of i giving the answer*

$$6^{10} = 2^{10}3^{10}.$$

A–2 Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

Solution: *The problem is evidently scale invariant so we take the side of the triangle on which R lies on the x-axis running from 0 to 1. Put the vertex at (x, y) with $y > 0$ and $0 < x < 1$ to make the triangle acute. We take the combined height of R and S to be λy with $0 < \lambda < 1$ and the height of triangle R to be $\lambda\theta y$ for some $0 < \theta < 1$.*

The corners of R are at

$$(\lambda\theta x, 0); (\lambda\theta x, \lambda\theta y); (1 - \lambda\theta(1 - x), 0); (1 - \lambda\theta(1 - x), \lambda\theta y).$$

The area is then

$$A(R) = (1 - \lambda\theta)\lambda\theta y$$

The corners of S are at

$$(\lambda x, \lambda\theta y); (\lambda x, \lambda y); (1 - \lambda(1 - x), \lambda\theta y); (1 - \lambda(1 - x), \lambda y).$$

and then

$$A(S) = \lambda(1 - \lambda)(1 - \theta)y$$

Finally

$$A(T) = y/2.$$

This makes

$$\frac{A(S) + A(R)}{A(T)} = 2\left\{(1 - \lambda\theta)\lambda\theta + \lambda(1 - \lambda)(1 - \theta)\right\}$$

which simplifies to

$$2(-\lambda^2\theta^2 + \lambda^2\theta + \lambda - \lambda^2).$$

At $\lambda = 0$ this ratio vanishes. The derivative with respect to θ is

$$2\lambda^2 - 4\lambda^2\theta$$

which vanishes for $\lambda > 0$ at, and only at, $\theta = 1/2$. The second θ derivative is negative so that $\theta = 1/2$ maximizes the ratio for each $\lambda > 0$. At $\theta = 1/2$ the ratio becomes

$$2(\lambda - 3\lambda^2/4) = 2/3 - 3(\lambda - 2/3)^2/2.$$

This is evidently maximized at $\lambda = 2/3$ and the maximum value is $2/3$.

A–3 Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m,$$
$$a_m(j+1) = (a_m(j))^2 + 2a_m(j), \qquad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

Solution: *Let $b_m(j) = a_m(j) + 1$ so that*

$$b_m(j+1) - 1 = (b_m(j) - 1)^2 + 2(b_m(j) - 1)$$

or

$$b_m(j+1) = b_m(j)^2 = b_m(j-1)^4 = \cdots.$$

This makes $b_m(m) = b_m(0)^{2^m}$. Since $b_m(0) = 1 + d/2^m$ we get

$$a_m(m) = b_m(m) - 1 = (1 + d/2^m)^{2^m} - 1 \to e^d - 1.$$

A–4 Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

Solution: *We begin by computing some powers of 3 modulo 100 to discover that*

$$3^{20} \equiv 1 \mod 100$$

The powers 3^j for j from 1 to 20 are, modulo 100,

$$3, 9, 27, 81, 43, 29, 87, 61, 83, 49, 47, 41, 23, 69, 7, 21, 63, 89, 67, 1$$

Any integer n may be written in the form

$$n = 20j + 4k + l$$

where $4k + l$ is the residue class of n modulo 20 and l the residue class of n modulo 4 (so that $l \in \{0, 1, 2, 3\}$ and $k \in \{0, 1, 2, 3, 4\}$). In this case

$$3^n \equiv 3^{4k+l} \mod 100.$$

Next note that

$$3^4 \equiv 1 \mod 20$$

from which it follows that

$$3^{4k} \equiv 1 \mod 20.$$

Let j_i, k_i, l_i satisfy

$$a_i = 20j_i + 4k_i + l_i$$

as above. Then

$$a_{i+1} \equiv 3^{l_i} \mod 20$$

We have $l_1 = 3$. *If* $l_i = 3$ *then*

$$3^{l_i} = 27 \equiv 7 \mod 20$$

which makes $l_{i+1} = 3$. *Thus* $l_i = 3$ *for all* i. *Next*

$$a_{i+1} \equiv 3^{4k_i + l_i} \mod 100$$

We have $k_1 = 0$, $k_1 = 1$. *If* $k_i = 1$ *then*

$$3^{4k_i + l_i} = 3^7 \equiv 87 \mod 100$$

so that $k_{i+1} = 1$. *Thus we have, for all* $i \geq 2$ *that* $k_i = 1$ *and* $l_i = 3$. *The residue class of* a_{i+1} *modulo 100 is then that of* 3^7 *which is 87. Thus only 87 occurs infinitely often as the last two digits of* a_i *– in fact 87 is the last two digits of* a_i *for all* $i \geq 3$.

A–5 Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1 \leq m \leq 10$ is $I_m \neq 0$?

Solution: *I claim the integral is non-zero only for* $m \in \{3, 4, 7, 8\}$ *(and in general for* m *congruent to 0 or 3 mod 4). Write* $\cos(jx) = (\exp(jix) + \exp(-jix))/2$ *and put*

$$f_m(x) = \prod_1^m \cos(jx) = 2^{-m} \prod_1^m \{(\exp(jix) + \exp(-jix)\}\,.$$

Let I *denote generic subset of* $\{1, \ldots, m\}$ *and* $\bar{I}$ *its complement. Then the product in* f_m *may be expanded to get*

$$I_m = 2^{-m} \sum_I \int_0^{2\pi} \exp\{i\{\sum_{j\in I} j - \sum_{j\in\bar{I}} j\}x\}dx.$$

The integral is 0 unless

$$\sum_{j\in I} j - \sum_{j\in\bar{I}} j = 0.$$

But the two sums in this formula add up to $m(m+1)/2$ *so the difference is actually*

$$2\sum_{j\in I} j - m(m+1)/2.$$

We thus have

$$\int f_m(x)dx = 2^{-m} \sum_{I:\sum_{j\in I} j = m(m+1)/4} 2\pi.$$

This is 0 only if there are no subsets I *satisfying the criterion. For* $m \in \{1, 2, 5, 6, 9, 10\}$ *(or in general* m *congruent to 1 or 2 modulo 4) the quantity* $m(m+1)/4$ *is not an integer so no such* I *exists and* $I_m = 0$. *For* $m = 3$ *take* $I = \{1, 2\}$. *For* $m = 4$ *take* $I = \{1, 4\}$. *For* $m = 7$ *take* $I = \{1, 2, 4, 7\}$ *and for* $m = 8$ *take* $I = \{1, 2, 7, 8\}$. *The examples show that the integral is positive in each case.*

A–6 If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial with real coefficients a_i, then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $F(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

(i) $g(0) = 1$, and

(ii) $\Gamma(F(x)^n) = \Gamma(g(x)^n)$

for every integer $n \geq 1$.

Solution: *I remark that I don't like the notation $\Gamma(p(x))$; I want $\Gamma(p)$ since the quantity in question is a functional of the function p not its value at x. Check that*

$$\Gamma(p) = \frac{1}{2\pi} \int_0^{2\pi} p\left(e^{ix}\right) p\left(e^{-ix}\right) dx$$

by writing the product as a double sum and integrating term by term. This means that

$$\Gamma(F^n) = \frac{1}{2\pi} \int_0^{2\pi} F^n\left(e^{ix}\right) F^n\left(e^{-ix}\right) dx$$

and we want this to be the same as

$$\Gamma(g^n) = \frac{1}{2\pi} \int_0^{2\pi} g^n\left(e^{ix}\right) g^n\left(e^{-ix}\right) dx$$

for a suitable polynomial g. Notice that if we can find a g satisfying (a) and such that $|z| = 1$ implies $|g(z)| = |F(z)|$ then we will be done since the integrands are just

$$|F\left(e^{ix}\right)|^{2n}$$

and

$$|g\left(e^{ix}\right)|^{2n}.$$

Now I will find a and b such that

$$g(x) = ax^2 + bx + 1$$

solves the problem. If $z = \exp ix$ then

$$|g(z)| = |a\cos(2x) + b\cos(x) + 1 + i(a\sin(2x) + b\sin(x))|$$

while

$$|F(z)| = |3\cos(2x) + 7\cos(x) + 2 + i(3\sin(2x) + 7\sin(x))|.$$

The equation $|F(z)| = |g(z)|$ becomes

$$\begin{aligned} &62 + 42(\sin(2\theta)\sin(\theta) + \cos(2\theta)\cos(\theta)) + 12\cos(2\theta) + 28\cos(\theta) \\ &\qquad = a^2 + b^2 + 1 + 2ab(\sin(2\theta)\sin(\theta) + \cos(2\theta)\cos(\theta)) + 2a\cos(2\theta) + 2b\cos(\theta). \end{aligned}$$

A standard trigonometric formula reduces this to

$$62 + 70\cos(\theta) + 12\cos(2\theta) = a^2 + b^2 + 1 + 2b(a+1)\cos(\theta) + 2a\cos(2\theta).$$

Take $a = 6$ to make the coefficients of $\cos(2\theta)$ match. Then $14b = 70$ gives $b = 5$ and finally

$$6^2 + 5^2 + 1 = 62$$

which establishes the identity. Thus

$$g(x) = 6x^2 + 5x + 1.$$

B–1 Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

Solution: *I claim $k = 3$ and that one set is $m_1 = 2$, $m_2 = 1$, $m_3 = 0$, $m_4 = -1$ and $m_5 = -2$. First*

$$(x-2)(x-1)x(x+1)(x+2) = x^5 - 5x^3 + 4x.$$

showing $k \leq 3$. If $p(x) = \prod_1^5 (x - m_i)$ has $k = 2$ then

$$p(x) = x^5 + ax^r$$

for some integers a and r with $0 \leq r \leq 4$. If $r \geq 2$ then $x = 0$ is a double root of $p(x)$ while if $r = 1$ the roots of p are 0 and the 4 roots of $-a$ in the complex plane – at most 2 of which can be integers. If $r = 0$ then the roots of p are the 5 fifth roots of $-a$ at least 3 of which are complex. And if $a = 0$ then $p(x) = x^5$ has a multiple root at 0.

B-2 Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Solution: *Define*

$$f_n(x) = (x+n)^n - n(x+n)^{n-1}.$$

Check that

$$f_n(0) = n^n - nn^{n-1} = 0$$

and

$$\frac{d}{dx} f_{n+1}(x) = (n+1)(x+n+1)^n - (n+1)n(x+n+1)^{n-1} = f_n(x+1).$$

Then

$$f_n(1) = (n+1)^n - n(n+1)^{n-1} = (n+1)^{n-1}(n+1-n) = (n+1)^{n-1}$$

so that

$$f_{100}(1) = 101^{99}$$

is the desired factorization.

B–3 Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

Solution: *Consider the pairs $M_j = 2^{K-j}$ and $N_j = 2^j$ for $j = 0, \ldots, K$ where K is a positive integer. Note that $M_j N_j = 2^K$. Let $C_j = \{(m,n) : m \leq M_j \text{ and } n \leq N_j\}$. Let $B_0 = C_0$ and for $j \geq 1$ put $B_j = C_j \setminus C_{j-1}$. Note that the cardinality of B_j is 2^{K-1} for $j \geq 1$ and 2^K for $j = 0$. Thus the cardinality of*

$$B = \cup_0^K B_j$$

is

$$2^K + K 2^{K-1}$$

If K is large enough so that

$$2^K + K2^{K-1} > 82^K$$

or

$$(1 + K/2) > 8$$

or

$$K > 14$$

then under the conditions of the theorem there must be a pair $(m, n) \in B$ with

$$a_{m,n} > 2^K \geq mn$$

B–4 Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

Solution: *Write $p = (\cos\theta, \sin\theta)$ with θ uniformly distributed on $(0, 2\pi)$. Let S be the rectangle with corners at $(\pm\cos\theta, \pm\sin\theta)$. Then R lies inside C if and only if $q \in S$. The area of S is $|4\cos\theta\sin\theta|$. The unit circle has area π so given θ the probability that R lies inside C is*

$$\frac{4|\cos\theta\sin\theta|}{\pi}.$$

Now average over θ to find that the desired probability is

$$\int_0^{2\pi} \frac{4|\cos\theta\sin\theta|}{\pi} d\theta/(2\pi).$$

This is just

$$\frac{8}{\pi^2}\int_0^{\pi/2} \cos\theta\sin\theta d\theta = \frac{4}{\pi^2}.$$

B–5 Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\, dt$. You may assume that $\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$.

Solution: *I haven't bothered to check all the algebra here. The method works but the algebra might be defective. Define*

$$G_n(\theta) = \int_0^\infty t^{-n/2} e^{-t} e^{-\theta/t} dt.$$

Note that

$$G_1(0) = \Gamma(1/2) = \sqrt{\pi};$$

this is a consequence via a change of variables of the hint given. Differentiate G_1 with respect to θ under the integral sign to see that for $\theta > 0$ (needed to justify an application of the dominated convergence theorem) we have

$$\frac{\partial G}{\partial\theta} = \theta G_5(\theta).$$

Then in G_5 integrate by parts taking $dv = \theta^{-1} t^{-2}\exp(-\theta/t)$ and $u = \exp(-t)$ to get

$$\frac{\partial G}{\partial\theta} = -G_1(\theta) - 2^{-1}G_3(\theta).$$

Then make the substitution $u = \theta/t$ in the integral defining G_3. This shows

$$G_1(\theta) = \sqrt{\theta} G_3(\theta).$$

This gives

$$\frac{\partial G}{\partial \theta} = -(1 + (2\sqrt{\theta})^{-1}) G_1.$$

In turn we see

$$\frac{\partial \log G}{\partial \theta} = -(1 + (2\sqrt{\theta})^{-1}).$$

This gives

$$\log G_1(\theta) = -\theta + \sqrt{\theta} + \log G(0)$$

or

$$G_1(\theta) = \sqrt{\pi} \exp\{\sqrt{\theta} - \theta\}.$$

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Finally: if we substitute $u = 1985t$ in the integral in the question we get

$$\sqrt{1985} G_1(1985^2) = \sqrt{1985\pi} \exp\{1985 - 1985^2\}.$$

Technical points. The Dominated Convergence Theorem applies only for $\theta > 0$ to permit differentiation under the integral sign. Having integrated the resulting differential equation for $\theta > 0$, however, we are entitled to evaluate at $\theta = 0$ because G_1 is continuous at $\theta = 0$ as may be verified by the DCT again.

B–6 Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.

Solution: *If M, M_1 and M_2 are in the group G then $MM_1 = MM_2$ implies $M_1 = M_2$ because M is invertible. Thus $\{MM_1, \ldots, MM_r\} = G$. If $A = \sum_{i=1}^r M_i/r$ then we see that $MA = A$ for all $M \in G$ and so $A^2 = A$. If λ is an eigenvalue of A with non-zero eigenvector v then $A^2 v = Av$ so $\lambda^2 v = \lambda v$ so $\lambda(1 - \lambda) = 0$. That is each eigenvalue of A is either 1 or 0. But the trace of any matrix is the sum of the roots of its characteristic polynomial. Since each such root is an eigenvalue we see that the trace of A is the multiplicity of 1 as a characteristic value of A. Since the sum is 0 the characteristic polynomial of A is λ^n.*

Now from $AA = A$ we learn that each non-zero column of A is an eigenvector of A for the eigenvalue 1. But there are no such eigenvectors so every column of A is 0.