The Fiftieth Annual William Lowell Putnam Competition
Saturday, December 2, 1989

All done

A–1 How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

Solution: *Such a number has the form*

$$\sum_{k=0}^{n} 10^{2k} = \frac{10^{2(n+1)}-1}{99} = \frac{10^{n+1}-1}{99} \times (10^{n+1}+1) = \frac{\sum_{k=0}^{n} 10^k}{11} \times (10^{n+1}+1).$$

The first term is an integer if n is odd since in that case the sum in the numerator consists of $(n+1)/2$ concatenated 11s. Thus if n is odd and the first term is not 1 our number is not prime. For $n > 1$ the sum in the numerator is more than 11. For $n = 1$ the first term is 1 and our number is 101 which is prime. If n is even then $10^{n+1}+1$ is divisible by 11 and, for $n \geq 2$ both $\sum_{k=0}^{n} 10^k$ and $(10^{n+1}+1)/11$ exceed 1 showing our number is not prime. Finally for $n = 0$ our number is 1 which is not prime. Thus only the number 101 is a prime of the desired form.

A–2 Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$ where a and b are positive.

Solution: *Put $u = bx$ and $v = by$ to get*

$$I \equiv \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx = ab \int_0^{ab} \int_0^{ab} exp\{\max(u^2, v^2)\} dv du.$$

It is easily checked that the integrals over $v < u$ and over $v > u$ are equal so

$$I = 2\int_0^{ab} \int_0^{u} e^{u^2} dv du = \int_0^{ab} 2u e^{u^2} du = e^{u^2}\Big|_0^{ab} = e^{a^2b^2} - 1.$$

A–3 Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

Solution: *Put $iu = z$ to see that the given polynomial is 0 at z if and only if*

$$f(u) \equiv 11u^{10} + 10u^9 + 10u + 11 = 0.$$

We may write

$$f(u) = u^5 p(u + 1/u)$$

where $p(u)$ is a polynomial of degree 5 found by comparing coefficients:

$$p(z) = 11(z^5 - 5z^3 + 5z) + 10(z^4 - 4z^2 + 2)$$

(Notice that $x + 1/x$ raised to an odd power contains only odd powers of x and raised to an even power contains only even powers of x. The absence of powers x^8 down to x^2 allows determination of the coefficients relatively easily.) If $p(z) = 0$ and $x + 1/x = z$ then $f(x) = 0$. The equation

$$x + 1/x = z$$

has roots

$$\frac{z \pm \sqrt{z^2 - 4}}{2}$$

If z is a real root of p with $z^2 \leq 4$ then the two roots of $x + 1/x = z$ lie on the unit circle. We will prove that p has 5 real roots and that all of these roots have $z^2 < 4$.

The polynomial p has the structure

$$p(u) = 11uq(u^2) + 10r(u^2)$$

where

$$q(x) = x^2 - 5x + 5$$

and

$$r(x) = x^2 - 4x + 2.$$

The roots of q are $q_1 = (5 - \sqrt{5})/2$ and $q_2 = (5 + \sqrt{5})/2$ while those of r are $r_1 = 2 - \sqrt{2}$ and $r_2 = 2 + \sqrt{2}$. Check that

$$r_1 < q_1 < r_2 < q_2.$$

If $z > 0$ and

$$q_1 \leq z^2 \leq r_2$$

then

$$q(z^2) < 0$$

and

$$r(z^2) < 0$$

so

$$p(z) < 0.$$

If $z > 0$ and $z < r_1$ then $p(z) > 0$ and if $z > 0$ and $z^2 > q_2$ then $q(z^2) > 0$ and $r(z^2) > 0$ so $p(z) > 0$. Thus p has a root in the range $\sqrt{r_1} < z < \sqrt{q_1}$ and another in the range $\sqrt{r_2} < z < \sqrt{q_2}$.

Now we look for negative roots. Notice that $p(-1) = -21 < 0$ so that there is a root between -1 and 0. Notice that if $q(z^2) < 0$ and $r(z^2) > 0$ and $z < 0$ then $p(z) > 0$. If $z < 0$ and $r_2 < z^2 < q_2$ then all these conditions are met. Note that $p(-2) = -2 < 0$. Finally

$$-2 < -\sqrt{r_2} < -1 < 0$$

and $p(-2) < 0$, $p(-\sqrt{r_2}) > 0$, $p(-1) < 0$ and $p(0) > 0$ so there are three negative roots, one in each of the ranges

$$-2 < z < -\sqrt{r_2}$$

$$-\sqrt{r_2} < z < -1$$

and

$$-1 < z < 0.$$

We have now demonstrated that p has 5 real roots in the interval $(-2, q_2)$ and since $q_2 < 2$ all roots, z, of $p(z)$ have $|z| < 2$. Thus all 10 roots of f are on the unit circle.

A–4 If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

Solution: *Yes. Write $\alpha = 0.x_1x_2x_3\cdots$ where the x_i are the binary expansion of α. For binary rationals α choose, for definiteness, the expansion which ends in infinitely many 0s. Now toss the fair coin recording X_i as 1 or 0 depending on whether the coin lands heads or tails on toss i. Player 1 wins if and only if the number $U \equiv 0.X_1X_2X_3\cdots$ is smaller than α. It is straightforward to check that U has a uniform distribution on the unit interval so that the probability that player 1 wins is α.*

A–5 Let m be a positive integer and let $\mathcal{G}$ be a regular $(2m+1)$-gon inscribed in the unit circle. Show that there is a positive constant A, independent of m, with the following property. For any points p inside $\mathcal{G}$ there are two distinct vertices v_1 and v_2 of $\mathcal{G}$ such that

$$\left|\,|p - v_1| - |p - v_2|\,\right| < \frac{1}{m} - \frac{A}{m^3}.$$

Here $|s - t|$ denotes the distance between the points s and t.

Solution: *Fix P in the polygon and let*

$$r_1 \le r_2 \le \cdots \le r_{2m+1}$$

be the $2m+1$ distances from P to the vertices sorted into increasing order. I claim that there is an A such that

$$r_{2m+1} < 2 - 2A/m^2.$$

If so then

$$0 \le r_{2m+1} - r_1 = \sum_{i=1}^{2m} (r_{i+1} - r_i) < 2(1 - A/m^2).$$

The average of the $2m$ numbers $r_{i+1} - r_i$ is therefore less than $1 - A/m^3$ and if follows that there is an i with $r_{i+1} - r_i < 1 - A/m^3$ which would finish the problem since r_i and r_{i+1} are distances from P to 2 distinct vertices.

The distance between points inside the polygon is maximized by two vertices of the polygon. (For instance the distance from a point P to point P' is always increased by moving P' to the boundary along the line joining P to P' and then P to the opposite boundary. Thus the maximum is clearly achieved by two points on the boundary. If P is on the boundary and P' is on some edge between two vertices the distance from P to P' is always maximized by moving P' to one of the two vertices. Repeat the argument to see that both points much be vertices.) Suppose then that A and B are two consecutive vertices of $\mathcal{G}$ as we proceed counterclockwise around the circle. The perpendicular bisector of AB passes through a vertex C on the other side of the triangle and the mid-point M on the segment AB. The distance from C to a point P on the circle increases as P moves away from C; thus AC and BC maximize the distance between two points in the polygon. It remains to compute that distance. If O is the origin of the circle then the angle AOB is $2\pi/(2m+1)$ and so the angle ACB is $\pi/(2m+1)$. The triangle CMA is a right triangle with the angle at C being $\pi/(4m+2)$. If D is the point opposite C on the circle then CAD is a right triangle with hypotenuse CD of length 2 and angle $\pi/(4m+2)$ so $|AC| = 2\cos(\pi/(4m+2))$.

It remains to analyze that cosine. I claim there is an $a > 0$ such that

$$\cos(x) \le 1 - ax^2$$

for all $0 < x \le \pi/6$; when $m = 1$ the argument of the cosine above is $\pi/6$ so $0 < \pi/(4m+2) \le \pi/6$. If so then

$$|AC| \le 2 - 2a\pi^2/(4m+2)^2 \le 2(1 - a\pi^2/(16m^2)).$$

This finishes the problem with $A = a\pi^2/16$. The function $f(x) = 1 - ax^2 - \cos(x)$ has $f(0) = f'(0) = 0$ and $f''(x) = \cos(x) - 2a$. If we take $2a = \cos(\pi/6) = \sqrt{3}/2$ then $f''(x) \ge 0$ for all $0 < x < \pi/6$. We then find f increasing on the given interval and so

$$f(x) \ge 0$$

for all $0 \le x \le \pi/6$. This finishes the problem with $a = \sqrt{3}/4$ and $A = \sqrt{3}\pi^2/64$, apparently.

A–6 Let $\alpha = 1 + a_1 x + a_2 x^2 + \cdots$ be a formal power series with coefficients in the field of two elements. Let

$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block} \\ 0 & \text{otherwise.} \end{cases}$$

(For example, $a_{36} = 1$ because $36 = 100100_2$ and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + x\alpha + 1 = 0$.

Solution: *The constant term is 1+1=0. Formally the coefficient of x^m is, for $m > 0$*

$$a_{m-1} + \sum_{ijk:i+j+k=m} a_i a_j a_k.$$

If i, j, and k are all different then $a_i a_j a_k$ occurs 6 times in any sum and so is 0 mod 2. The terms $a_i^2 a_j$ with $i \neq j$ occur three times for each i, j pair with $2i + j = m$ and $i \neq j$. If $m = 3i$ then the term $a_i^3 = a_i^2 a_i$ occurs one time. Note that $a_i^2 = a_i$. We now see that the coefficient of x^m is

$$b_m \equiv a_{m-1} - \sum_{i:0 \leq 2i \leq m} a_i a_{m-2i}.$$

If we stick a 1 on the end of the binary representation of k then we do not change the length of any sequence of consecutive 0s. The act of concatenating a 1 corresponds to writing the binary representation of $2k + 1$. Thus

$$a_{2k+1} = a_k.$$

The act of concatenating the sequence 10 to the end of a binary sequence always creates a sequence of 1 zero, at the end so for all k we have

$$a_{4k+2} = 0.$$

Adding 2 zeros to the end of a binary does not change the parity of any stretch of zeros so for all $k > 0$ we have

$$a_{4k} = a_k.$$

Notice that

$$a_{4k+1} = a_{2k}$$

and

$$a_{4k+3} = a_{2k+1} = a_k.$$

A given m belongs to 1 of 4 residue classes modulo 4. Consider first the case $m = 4k+1$. We must compute

$$b_{4k+1} = a_{4k} - \sum_{i:0 \leq i, 2i \leq 4k+1} a_i a_{4k+1-2i}.$$

There are $2k + 1$ terms in the sum and we pair them up: first with last, second with second last, and so on with the middle term, $i = k$, left over to get

$$\begin{aligned} b_{4k+1} &= a_{4k} - a_k + \sum_{i=0}^{k-1} a_i a_{2(2k-i)+1} + \sum_{j=0}^{k-1} a_{2k-j} a_{4k+1-2(2k-j)} \\ &= a_{4k} - a_k + \sum_{i=0}^{k-1} \left\{ a_i a_{2(2k-i)+1} + a_{2k-i} a_{2i+1} \right\}. \end{aligned}$$

Since $a_{4k} = a_k$, $a_{2(2k-i)+1} = a_{2k-i}$, *and* $a_{2i+1} = a_i$ *we see that each term is 0.*

Now look at $m = 2k + 3$ *and pair* $a_i a_{4k+3-2i}$ *with* $a_{2k+1-i}a_{4k+3-2(2k+1-i)} = a_{2k+1-i}a_{2i+1}$. *Since* $m = 4k + 3$ *we have* $m - 1 = 4k + 2$ *which is congruent to 2 modulo 4 so* $a_{m-1} = 0$. *Use* $a_{2i+1} = a_i$ *and* $a_{4k+3-2i} = a_{2k+1-i}$ *to see that the sum in any pair is 0. The sum consists of* $2k + 2$ *terms so* $k + 1$ *pairs all of which add to 0.*

Next we do $m = 4k$ *and note that if* i *is odd then* a_{4k-2i} *is 0 since* $4k - 2i$ *is congruent to 2 mod 4. For* i *even only* i *divisible by 4 can have* a_i *non-zero. The sum reduces to*

$$\begin{aligned} b_{4k} &= a_{4k-1} + \sum_i a_{4i}a_{4k-8i}1(0 \le 8i \le 4k) \\ &= a_{4(k-1)+3} + \sum_i a_i a_{k-2i}1(0 \le 2i \le k) \\ &= a_{k-1} + \sum_i a_i a_{k-2i}1(0 \le 2i \le k) \\ &= b_k \end{aligned}$$

This is precisely the coefficient of x^k.

Finally we consider $m = 4k + 2$. *Since* $m - 1 = 4k + 1$ *we see* $a_{m-1} = a_{2k}$. *The remaining terms in the sum have the form* $a_i a_{4k+2-2i}$. *For* i *even the subscript* $4k + 2 - 2i$ *is congruent to 2 mod 4 so the term is 0. If* i *is odd, say* $i = 2j + 1$ *then* $a_i = a_j$ *and* $a_{4k+2-2i} = a_{4k+2-2(2j+1)} = a_{4(k-j)} = a_{k-j}$ *so the terms are* $a_j a_{k-j}$. *We are summing over odd* i *such that* $0 \le i \le 4k + 2$ *or* j *such that* $0 \le 2j + 1 \le 4k + 2$ *which is just* $0 \le j \le k + 1$. *Thus*

$$\begin{aligned} b_{4k+2} &= a_{2k} + \sum_{j=0}^{k+1} a_j a_{k-j} \\ &= a_{2k} + \sum_{j=0}^{k+1} a_j a_{2(k-j)+1} \\ &= a_{2k} + \sum_{j=0}^{k+1} a_j a_{2k+1-2j} \\ &= b_{2k+1} \end{aligned}$$

The proof now follows by induction. If the identity is wrong there is a least m *for which* $b_m = 1$ *mod 2. Check that* $b_0 = b_1 = b_2 = b_3 = 0$ *by direct computation. The 4 previous paragraphs show that* m *cannot be congruent to 1 or 3 mod 4 and that if* m *is congruent to 0 mod 4 then* $b_m = 1$ *implies* $b_k = 1$ *where* $m = 4k$. *Since* $k < m$ *this is impossible. The same argument works,* mutatis mutandis, *for* m *congruent to 2 mod 4.*

I feel that this proof is right but somehow misses the point. How could anyone have discovered this arcane identity? I think the way it was discovered likely provides a proof with more understanding.

B–1 A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\dfrac{a\sqrt{b} + c}{d}$, where a, b, c, d are integers.

Solution: *The square might as well have vertices at* $(\pm 1, \pm 1)$. *Divide the square into 4 wedges with boundaries the lines* $y = \pm x$. *For an* (x, y) *with* $y > |x|$, *which is the top wedge, the closest edge is* $y = 1$ *and the distance to the edge is* $1 - y$ *while the distance to the center is* $\sqrt{x^2 + y^2}$. *A point* (x, y) *with*

$$1 - y > \sqrt{x^2 + y^2} \text{ and } y > |x|$$

must have

$$1 - 2y + y^2 > x^2 + y^2 \text{ or } y < \frac{1 - x^2}{2}$$

and the area of the set of points with $|x| < y < (1-x^2)/2$ *is*

$$2\int_0^{x^*} \left\{(1-x^2)/2 - x\right\} dx = x^* - (x^*)^3/3 - (x^*)^2$$

where x^* *is the positive root of* $(1-x^2)/2 = x$. *The roots are*

$$\frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

so $x^* = \sqrt{2} - 1$. *The total area of the square is 4 and the area of one wedge is 1. The area in the wedge closer to the center is*

$$\sqrt{2} - 1 - (\sqrt{2}-1)^3/3 - (\sqrt{2}-1)^2 = (1 - 2/3 - 1 + 2)\sqrt{2} - 1 + 2 + 1/3 - 2 - 1 = \frac{4\sqrt{2}-5}{3}$$

which is the desired probability up to algebra.

B–2 Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

Solution: *Yes. Fix any* a *and find the smallest* n *such that* $a^n = a^m$ *for some* $m < n$. *Cancel* $m-1$ *factors of* a *to find* $a = a^{n+1-m}$. *Define* $\mathbf{1} = a^{n-m}$. *I need only show that* $\mathbf{1}$ *is a left and right identity and then establish the existence of inverses. We have*

$$a^{n+1-m}x = ax$$

for each x. *Cancel one* a *to see that* $\mathbf{1}$ *is a left identity. Similarly it is a right identity. If* x *is any element of* S *the same argument shows that there is a power of* x *which we call* $\mathbf{1}'$ *which is also a left and right identity. But then*

$$\mathbf{1}' = \mathbf{1}'\mathbf{1} = \mathbf{1}.$$

This shows that identities are unique and that x *has an inverse.*

B–3 Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \le e^{-\sqrt{x}}$ for $x \ge 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\, dx$$

(sometimes called the nth moment of f).

a) Express μ_n in terms of μ_0.

b) Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

Solution: *The condition on* f *shows that we may integrate by parts to get*

$$\begin{aligned}
\mu_n &= \frac{x^{n+1}}{n+1} f(x)\Big|_0^\infty - \frac{1}{n+1}\int_0^\infty x^{n+1} f'(x) dx \\
&= \frac{3}{n+1}\int_0^\infty x^{n+1} f(x) dx - \frac{6}{n+1}\int_0^\infty x^{n+1} f(2x) dx \\
&= \frac{3\mu_{n+1}}{n+1} - \frac{6}{(n+1)2^{n+2}}\int_0^\infty u^{n+1} f(u) du \\
&= 3\frac{1 - 2^{-(n+1)}}{n+1}\mu_{n+1}
\end{aligned}$$

It follows that

$$\frac{\mu_n}{\mu_0} = \prod_1^n \frac{\mu_j}{\mu_{j-1}} = \prod_1^n \frac{j}{3(1-2^{-j})} = \frac{n!}{3^n} \prod_1^n \frac{1}{1-2^{-j}}.$$

This does (a).

It follows then that

$$3^n \mu_n / n! = \mu_0 \prod_1^n \frac{1}{1-2^{-j}}.$$

To finish the problem we need only show the the product has a non-zero limit or that its log has a finite limit. But

$$\log\left(\prod_1^n \frac{1}{1-2^{-j}}\right) = -\sum_1^n \log(1-2^{-j})$$

It is easily checked that there is a constant A *such that for all* $0 \le x \le 1/2$ *we have* $0 \le -\log(1-x) \le Ax$. *Thus the logarithm of our product is bounded between 0 and*

$$A \sum_0^\infty 2^{-j} = A.$$

Since each term in the sum is positive the logarithm of the product increases with n *and so must have a limit. We have shown that*

$$\lim 3^n \mu_n / n! = \mu_0 \prod_1^\infty \frac{1}{1-2^{-j}}$$

and that the infinite product exists and is not 0. This does (b).

B–4 Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

Solution: *No. Let* C *be the countable set and* $\mathcal{C}$ *be such an uncountable collection. There are countably many finite subsets of* C *since there are countably many subsets of size* n *for each* n. *Let* $\mathcal{B}_A$ *be the collection of all pairs* B_1, B_2 *of members of* $\mathcal{C}$ *with* $B_1 \cap B_2 = A$. *The union over* A *of* $\mathcal{B}_A$ *is* $\mathcal{C} \times \mathcal{C} \setminus \{(B, B) : B \in \mathcal{C}\}$. *This set is uncountable so there is a fixed finite subset* A *of* C *such that* $\mathcal{B}_A$ *is uncountable. Remove* A *from each set to see that there is an uncountable family of subsets of a countable set, namely* $C \setminus A$, *which is pairwise disjoint. So now* C *is a countable set and* $\mathcal{C}$ *an uncountable family of pairwise disjoint subsets of* C. *For each integer* n *at most one element of* $\mathcal{C}$ *contains* n. *So the cardinality of* $\mathcal{C}$ *must be countable (finite or countably infinite).*

B–5 Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2, and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of T, and O is the center of the circle. Determine the least upper bound of $\frac{s_1 - s_2}{d}$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained.

Solution: *Orient the circle with the* AB *and* CD *parallel to the* x*-axis. Suppose* AB *is the intersection of* $y = y_1$ *with the disk and* CD *is the intersection of* $y = y_2$ *with the disk with* $-1 < y_1 < y_2 < 1$. *So* $A = (-x_1, y_1)$, $B = (x_1, y_1)$, $C = (x_2, y_2)$ *and* $D = (-x_2, y_2)$ *with* $x_1 > 0$ *and* $x_2 > 0$. *Note that* $x_i^2 + y_i^2 = 1$ *so that* $x_i = \sqrt{1 - y_i^2}$. *The diagonals intersect at the point* $E = (0, t)$. *If* $t = 0$ *then* AC *and* BD *are diagonals of the circle so the angles at* B *and* D *are right angles. Then* $ABCD$ *is a square. Notice that the slope of* AE *is the same as that of* AC *so that*

$$\frac{y_2 - y_1}{x_2 + x_1} = \frac{t - y_1}{x_1}.$$

The length of AB is $s_1 = 2x_1$ *and that of* CD *is* $s_2 = 2x_2$ *so we are to maximize the quantity*

$$\frac{2x_1 - 2x_2}{|t|}.$$

This is negative unless $x_1 \geq x_2$ *which implies* $|y_1| \leq |y_2|$. *If* $x_1 = x_2$ *then of course either* $y_1 = -y_2$ *and* $ABCD$ *is a square so that* $t = 0$ *or*$AB = CD$ *and our ratio is 0. We therefore assume that* $|y_1| < |y_2|$. *If* $y_2 < 0$ *this means that the segment* AB *lies above* CD *contradicting our counterclockwise orientation. So we assume* $-y_2 < y_1 < y_2$ *with* $y_2 > 0$. *It is easy to check that in this situation* $t > 0$.

For a given value of t *once the vertex* A *has been placed the other 3 vertices are determined. Moreover each value of* y_1 *corresponds to a single* A. *For a fixed* $t > 0$ *consider the slope of the segment* AC. *As* y_1 *increases (with* t *fixed) the slope of* AE, *namely,*

$$\frac{t - y_1}{\sqrt{1 - y_1^2}}$$

decreases. (Its derivative is

$$-\frac{1}{\sqrt{1-y_1^2}} + \frac{y_1(t-y_1)}{(1-y_1^2)^{3/2}} = \frac{-(1-y_1^2) + y_1 t - y_1^2}{(1-y_1^2)^{3/2}} = -\frac{1 - ty_1}{(1-y_1^2)^{3/2}}$$

which is negative since $|t| < 1$ *and* $|y_1| < 1$.*) It follows that* y_2 *is a decreasing function of* y_1 *on the interval* $0 < y_1 < t$ *and an increasing function of* y_1 *on the interval* $-y_2 < y_1 < 0$. *Then* x_2 *is an increasing function of* y_1 *on* $0 < y_1 < t$ *and* x_1 *is a decreasing function of* y_1 *on the same interval so*

$$\frac{2(x_1 - x_2)}{t}$$

is a decreasing function of y_1 *on the interval* $0 < y_1 < t$. *On the interval* $-y_2 < y_1 < 0$ *we have* y_2 *is an increasing function of* y_1 *and* x_2 *is a decreasing function of* y_1. *Since* $x_1 = \sqrt{1 - y_1^2}$ *it is an increasing function of* y_1 *over the same interval. For a fixed* t *then, the quantity* $x_2 - x_1$ *is an increasing function of* y_1 *on the interval* $-y_2 < y_1 < 0$ *and a decreasing function of* y_1 *on the interval* $0 < y_1 < t$. *Thus the ratio* $2(x_2 - x_1)/t$ *is maximized over* $y_2 \leq y_1 \leq t$ *at* $y_1 = 0$.

Now at $y_1 = 0$ *we find* $x_1 = 1$ *and*

$$\frac{y_2}{1 + x_2} = t.$$

Since $x_2^2 + y_2^2 = 1$ *we find* $x_2^2(1 + t^2) = 1$ *so our target ratio is*

$$\frac{2(1 - x_2)}{t} = \frac{2(1 - x_2^2)}{t(1 + x_2)}.$$

But $t(1 + x_2) = y_2$ *and* $1 - x^2 = y_2^2$ *so our ratio is just*

$$2y_2.$$

This latter quantity is maximized at $y_2 = 1$ *and the maximum value is 2. It is achieved by letting* $t \to 1$ *and* $x_2 \to 0$ *which corresponds to* AB *being a diameter of the circle and* $C = D$ *with* CO *being the perpendicular bisector of* AB. *In other words. The maximum is achieved by a right triangle with hypotenuse* AB *being a diameter and the third vertex half way between* A *and* B.

B–6 Let $(x_1, x_2, \ldots x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt$, where P is a polynomial of degree n, independent of f, with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1$.

Solution: *We are discussing (since $f(1) = 0$)*

$$\mathrm{E}\left\{\sum_{0}^{n-1}(X_{(i+1)} - X_{(i)})f(X_{(i+1)})\right\}$$

where $X_{(1)} < \cdots < X_{(n)}$ are the order statistics of an independent and identically distributed sample $X_1, \ldots, X_n$ from the Uniform[0,1] distribution and where $X_{(0)} = 0$ and $X_{(n+1)} = 1$. These uniform order statistics have many well known properties. For instance

$$\frac{X_{(1)}}{X_{(i+1)}}, \ldots, \frac{X_{(i)}}{X_{(i+1)}}$$

are uniformly distributed over the set

$$\{(y_1, \ldots, y_i) \in R^i : 0 < y_1 < \cdots < y_i < 1\};$$

moreover this vector is independent of $X_{(i+1)}$. The marginal density of $X_{(i)}$ is

$$f_i(x) = \frac{n!}{(i-1)!(n-i)!}x^{i-1}(1-x)^{n-i}1(0 < x < 1).$$

Let $Z_i = X_{(i)}/X_{(i+1)}$. Then

$$\begin{aligned}
\mathrm{E}\left\{\sum_{0}^{n}(X_{(i+1)} - X_{(i)})f(X_{(i+1)})\right\} &= \sum_{0}^{n-1}\mathrm{E}\left\{(1-Z_i)X_{(i+1)}f(X_{(i+1)})\right\} \\
&= \sum_{0}^{n-1}\mathrm{E}(1-Z_i)\mathrm{E}(X_{(i+1)}f(X_{(i+1)})) \\
&= \sum_{0}^{n}\frac{1}{i+1}\int_0^1 \frac{n!}{i!(n-i-1)!}x^{i+1}(1-x)^{n-i-1}f(x)dx
\end{aligned}$$

Define

$$P(t) = \sum_{0}^{n-1}\binom{n}{i+1}t^{i+1}(1-t)^{n-i-1} = \sum_{1}^{n}\binom{n}{i}t^i(1-t)^{n-i} = \sum_{0}^{n}\binom{n}{i}t^i(1-t)^{n-i} - (1-t)^n = 1-(1-t)^n$$

to finish the problem.

(Let $H(x)$ denote the sum with x being shorthand for $x_1, \ldots, x_n$. To try to do the problem without the machinery above we are asked to evaluate

$$n!\int_0^1\int_{x_1}^1\int_{x_2}^1\cdots\int_{x_{n-1}}^1 H(x)dx_n\cdots dx_1$$

(The $n!$ arises because the volume of the set of xs described is $1/n!$.) We do the integral of the ith term in the sum fixing x_i and x_{i+1} and integrating out the other terms first, then integrating over $0 < x_i < x_{i+1} < 1$. We can integrate in the order x_1 first, then x_2 and so on to x_{i-1} to get $x_i^{i-1}/(i-1)!$. Similarly integrate over x_n, x_{n-1} and so on to x_{i+2} to get $(1-x_{i+1})^{n-i-1}/(n-i-1)!$. This makes

$$n!\int_0^1\int_{x_1}^1\int_{x_2}^1\cdots\int_{x_{n-1}}^1 (x_{i+1}-x_i)f(x_{i+1})dx_n\cdots dx_1$$

simplify to

$$\frac{n!}{(i-1)!(n-i-1)!}\int_0^1\int_0^{x_{i+1}}(x_{i+1}-x_i)x_i^{i-1}(1-x_{i+1})^{n-i-1}dx_idx_{i+1}.$$

Change variables on the inside integral to $u = x_i/x_{i+1}$ *to get*

$$\frac{n!}{(i-1)!(n-i-1)!}\int_0^1\int_0^1 (1-u)u^{i-1}x_{i+1}^{i+1}(1-x_{i+1})^{n-i-1}du dx_{i+1}.$$

The inside integral is now $1/i - 1/(i+1) = 1/(i(i+1))$ *giving*

$$\frac{n!}{(i+1)!(n-i-1)!}\int_0^1 t^{i+1}(1-t)^{n-i-1}f(t)dt.$$

These calculations are valid for $1 \leq i < n-1$. *For* $i = 0$ *we have* $x_0 = 0$ *and do only the integrals over* x_n *to* x_2 *to get*

$$\frac{n!}{(n-1)!}\int_0^1 x_1(1-x_1)^{n-1}f(x_1)dx_1.$$

For $i = n-1$ *we do only integrals over* x_1 *to* x_{n-1} *to get*

$$\frac{n!}{(n-1)!}\int_0^1 x_n^{n-1}(1-x_n)f(x_n)dx_n.$$

Summing over i *gives the result where*

$$P(t) = \sum_{i=0}^{n-1}\frac{n!}{(i+1)!(n-i-1)!}t^{i+1}(1-t)^{n-i-1} = 1-(1-t)^n$$

by the binomial formula. This polynomial is clearly between 0 and 1. It looks to me that replacing $f(x)$ *by* $f(x)(1-x^m)$ *and letting* m *go to* ∞ *allows us to remove the hypothesis that* $f(1) = 0$.