VIRTUAL PATH LAYOUTS IN ATM NETWORKS *

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Abstract. We study virtual path layouts in a very popular type of fast interconnection networks, namely ATM networks. One of the main problems in such networks is to construct path layouts that minimize the hop-number (i.e. the number of virtual paths between any two nodes) as a function of the edge congestion c (i.e. the number of virtual paths going through a link). In this paper we construct for any n vertex network H and any c a virtual path layout with hop-number $O(\frac{diam(H)\log\Delta}{\log c})$, where $diam(H)$ is the diameter of the network H and Δ is its maximum degree. Involving a general lower bound from [6] we see that these hop-numbers are optimal for bounded degree networks with the diameter $O(\log n)$ for any congestion c. In the case of unbounded degree networks (with the diameter $O(\log n)$) these hop-numbers are optimal for any $c \geq \Delta$. For instance, this gives optimal hop-numbers for hypercube related networks. Moreover, we improve known results for paths and meshes and prove optimal hop-numbers for hypercubes.

Key words. ATM network, congestion, hop-number, virtual paths layout

AMS subject classifications. 68M10, 90B12

1. Introduction. Broadband Integrated Services Digital Network (B-ISDN) is a new paradigm in digital communication which integrates previous distinct networks (telephone, cable television, computer...) into a single digital network. The basic network transmission medium is a fiber-optic cable capable of transfering data at very high rates. The bottleneck caused by slow software-based switches is resolved by special purpose fast hardware. To utilize this, a new multiplexing and switching technology called ATM (Asynchronous Transfer Mode) was proposed (see e.g. [9]). Packet routing in ATM networks is based on relatively small fixed-size packets. Messages may be transmitted through arbitrarily long virtual paths. Packets are routed along these paths by maintaining a routing field whose subfields determine intermediate destinations of the packet, i.e. end-points of virtual paths on its way to final destination. One of the main problems in such networks is to construct path layout that minimizes the hop-number (i.e. the number of virtual paths between any two nodes) as a function of the edge congestion c (i.e. the number of virtual paths going through a link).

In this paper we construct for any n vertex network H and any c a virtual path layout with the hop-number $O(\frac{diam(H)\log\Delta}{\log c})$, where $diam(H)$ is the diameter of the network H and Δ is its maximum degree. Involving a general lower bound from [6] we see that these hop-numbers are optimal for bounded degree networks with the diameter $O(\log n)$ for any congestion c. In the case of unbounded degree networks (with the diameter $O(\log n)$) these hop-numbers are optimal for any $c \geq \Delta$. For instance, this gives optimal hop-numbers for hypercube related networks. Moreover, we improve known results for paths and meshes of Kranakis et al. [6]. Finally, we prove optimal hop-numbers for hypercubes.

1.1. Model, notation and results. The basic model of ATM networks was introduced by Gerstel et al. in [1, 2, 4, 5] and further developed in [3, 6]. By a network we understand any graph $G = (V_G, E_G)$. By $diam(G)$ we denote the *diameter* of G. Similarly, by $d(x, y)$ we denote the distance of the vertices x and y in G. Let Δ denote

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the maximum degree of a graph. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs such that $|V_G| \leq |V_H|$. An *embedding* of G in H is a pair of mappings (ϕ, ψ) satisfying

$$\phi : V_G \to V_H \quad \text{is an injection,} \quad \psi : E_G \to \{ \text{ set of all paths in } H \ \},$$

such that if $uv \in E_G$ then $\psi(uv)$ is a path between $\phi(u)$ and $\phi(v)$. Define the *congestion of an edge* $e \in E_H$ as $cg_{(\phi,\psi)}(e) = |\{f \in E_G \ : \ e \in \psi(f)\}|$ and the *congestion* of G in H as $cg(G, H) = \min_{(\phi,\psi)} \max_{e \in E_H} \{cg_{(\phi,\psi)}(e)\}$. Note that if there is no confusion we will omit the subscript (ϕ, ψ) in $cg_{(\phi,\psi)}(e)$.

The paths $\{\psi(E_G)\}$ are called *virtual paths*. The virtual path problem can be mathematically defined as: Given a graph H and a positive number c, among all graphs G satisfying $|V_G| = |V_H|$ and $cg(G, H) \leq c$, find a graph G_0 of minimum diameter. The minimum diameter is called the *hop-number* and denoted by $Hop_H(c)$.

Gerstel and Zaks [5] studied virtual path layouts for paths, cycles and meshes. They assumed an additional requirement that virtual channels are the shortest paths in the networks. Kranakis et al. [6] dropped the requirement and proved several optimal results on the hop-number of paths and meshes for small c. Namely, if P_n denotes an n-vertex path then $Hop_{P_n}(2) = \sqrt{2n} + o(1)$, $Hop_{P_n}(c) = \Theta(n^{1/c})$, when c is a constant and $Hop_{P_n}(\log^2 n/\log\log n) = \Theta(\log n/\log\log n)$, and if M_n denotes an $n \times n$ mesh then $Hop_{M_n}(c) = \Theta(\log n)$, for constant c.

We assume the same model as Kranakis et al. [6]. Our main result says that for any $c \geq 1$ and any n vertex graph H of bounded degree and diameter $O(\log n)$, $Hop_H(c) = \Theta(\log n/\log c)$. Several standard networks belong to this class of graphs, e.g. mesh of trees, Butterfly, cube-connected-cycles, binary de Bruijn or shuffle-exchange graph. If $c \geq \Delta$, then we construct optimal virtual path layouts for some unbounded degree networks, e.g. star, pancake and k-ary de Bruijn and Kautz graphs.

Further we improve the results of Kranakis et al. [6] in the following way:

(i) $Hop_{P_n}(c) = \Theta\left(\frac{\log n}{\log c}\right)$, if $c \geq \log^{1+\varepsilon} n$, for any fixed $\varepsilon > 0$.

(ii) If H denotes 2-dimensional $n \times n$ mesh then $Hop_H(c) = \Theta\left(\frac{\log n}{\log c}\right)$, for $c \geq 2$.

Finally we show that for the n-dimensional hypercube Q_n

$$Hop_{Q_n}(c) = \begin{cases} \Theta\left(\frac{n}{\log n}\right) & \text{if } 2 \leq c \leq n, \\ \Theta\left(\frac{n}{\log c}\right) & \text{otherwise.} \end{cases}$$

2. General Bounds . In this section we describe a general method for constructing virtual path layouts and then apply it to some standard networks. Since we deal with asymptotic results only, we can restrict ourselves to looking for the graph G_0 among trees:

PROPOSITION 2.1. *Let a graph G_0 minimize $diam(G)$ with respect to $cg(G, H) \leq c$. Then there exists a tree T with $diam(T) \leq 2diam(G_0)$ and $cg(T, H) \leq c$.*
Indeed, we can take T as a breadth first search spanning tree of H_0. We will often apply the following useful lemma from [10].

LEMMA 2.2. *Let G_1, G_2 and G_3 be graphs with $|V_{G_1}| \leq |V_{G_2}| \leq |V_{G_3}|$. Consider an embedding of G_1 into G_2 and an embedding of G_2 into G_3. Then*
$cg(G_1, G_3) \leq cg(G_1, G_2)cg(G_2, G_3)$.

It is worth noting that an embedding of G_1 into G_2 followed by the embedding of G_2 into G_3 induces an embedding of G_1 into G_3. An image of an edge under this composed embedding is not necessarily a path, but a walk in general. It is easy to transform the walk into a path by omitting multiple edges and vertices.

Throughout, we will use a general lower bound of Kranakis et al. [6].

LEMMA 2.3. *For any n vertex graph H of maximum degree Δ and for any $c \geq 1$*

$$Hop_H(c) \geq \frac{\log n}{\log(c\Delta)} - 1.$$

By $R(k,n)$ we denote the complete k-ary n-level tree. Note that by a k-ary 1-level and 0-ary k-level tree we mean a single vertex. We sometimes abbreviate this notation in the case of $R(2,n)$ to B_n (the complete binary tree) and $R(l,2)$ to S_l (the star on $l+1$ vertices).

Let T be a rooted tree. Let G be a rooted tree with $l \geq k$ leaves, where l and k are positive integers. By $G * kT$ we denote the graph constructed as follows: Take k copies of T and identify the roots of the copies of T with k distinct leaves of G.

For $d,i,n \geq 1$ and $k = d^i$ define the tree $F(k;d,n)$ recursively as follows. If $i = 1$ set $F(k;d,n) = R(d,n)$. If $i \geq n$ set $F(k;d,n) = R(\frac{d^n-1}{d-1} - 1, 2)$. If $1 < i < n$, set $F(k;d,n) = R(\frac{d^{i+1}-1}{d-1} - 1, 2) * kF(k;d,n-i)$. Note that the number of vertices of $F(k;d,n)$ is $\frac{d^n-1}{d-1}$. A picture of $F(4;2,6)$ is depicted in Fig. 1 (the left tree).

LEMMA 2.4. *For $d,i,n \geq 1$ and $k = d^i$, $diam(F(k;d,n)) = 2(a + \lceil \frac{r}{i} \rceil) = 2\lceil \frac{n-1}{i} \rceil$, where $n-1 = ai + r,\ 0 \leq r < i$.*

Proof. For $n = 1,2$, the Lemma is easy to observe. We assume that Lemma is true for all $l < n$ and we prove it for $l = n$. For $i = 1$, $diam(R(d,n)) = 2(n-1)$, the Lemma is true. Similarly, for $i \geq n$, $diam(R(\frac{d^n-1}{d-1} - 1, 2)) = 2$. Now, let $1 < i < n$. Hence $F(k;d,n) = R(\frac{d^{i+1}-1}{d-1} - 1, 2) * kF(k;d,n-i)$. Let $n-1 = ai+r,\ 0 \leq r < i$. Since $n-i \geq 1$ and since $n-i = (a-1)i+r$, by induction, $diam(F(k;d,n-i)) = 2(a-1+\lceil \frac{r}{i} \rceil)$, and $diam(R(\frac{d^{i+1}-1}{d-1} - 1, 2)) = 2$. It follows from the construction of $F(k;d,n)$ that $diam(F(k;d,n)) = diam(F(k;d,n-i)) + 2 = 2(a + \lceil \frac{r}{i} \rceil) = 2\lceil \frac{n-1}{i} \rceil$. □

LEMMA 2.5. *For $d,i,n \geq 1$ and $k = d^i$, $cg(F(k;d,n), R(d,n)) \leq \frac{k-1}{d-1}$.*

Proof. We proceed by induction on n. For $n = 1,2$, the result is easy to see. Assuming the result is true for $l \leq n-1$, we prove it for $l = n$. For $i = 1$, the statement of Lemma is true (we take as the mapping (ϕ,ψ) the natural isomorphism $\phi :\ F(k;d,n) \to R(d,n)$, and ψ the induced mapping by ϕ). If $i \geq n$, let ϕ map the root of $R(\frac{d^n-1}{d-1} - 1, 2)$ on the root of $R(d,n)$ and other vertices, injectively in an arbitrary way, on remaining vertices of $R(d,n)$. Further let ψ map each edge e of $R(\frac{d^n-1}{d-1} - 1, 2)$ on the unique paths in $R(d,n)$ connecting the images of the end-vertices of e. It is a matter of routine to observe that the congestion of each edge of $R(d,n)$ induced by (ϕ,ψ) is at most $\frac{k-1}{d-1}$ (the maximum congestion is achieved on edges incident with the root of $R(d,n)$), thus $cg(R(\frac{d^n-1}{d-1} - 1, 2), R(d,n)) \leq \frac{k-1}{d-1}$. Now, let $1 < i < n$. By definition $F(k;d,n) = R(\frac{d^{i+1}-1}{d-1} - 1, 2) * kF(k;d,n-i)$. It holds that $R(d,n) = R(d,i+1) * kR(d,n-i)$. Using induction we can embed $kF(k;d,n-i)$ into $kR(d,n-i)$ (each $F(k;d,n-i)$ into some $R(d,n-i)$) with congestion $\leq \frac{k-1}{d-1}$ and thus $cg(kF(k;d,n-i), kR(d,n-i)) \leq \frac{k-1}{d-1}$. Similarly, we obtain $cg(R(\frac{d^{i+1}-1}{d-1} - 1, 2), R(d,i+1)) \leq \frac{k-1}{d-1}$. Obviously, these two embeddings together induce an embedding (ϕ,ψ) of $F(k;d,n)$ into $R(d,n)$ with congestion at most $\frac{k-1}{d-1}$. □

REMARK 2.1. *The embedding (ϕ,ψ), constructed in the previous proof, has the property that for each vertex $v \in V_{F(k;d,n)}$ there is at most one its neighbour $u \in V_{F(k;d,n)}$ such that the unique $\phi(v) - \phi(u)$ path in $R(d,n)$ contains a vertex x with $d(\phi(x), r) < d(\phi(v), r)$, where r is the root of $R(d,n)$. This can be observed in Fig. 1, where the embedding of $F(4;2,6)$ in $R(2,6)$ is depicted.*

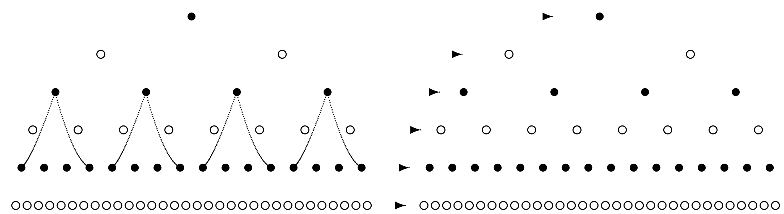

Fig. 1 The embedding of $F(4;2,6)$ in $R(2,6)$

LEMMA 2.6. *Let $\Delta \geq 3$, $i \geq 1$ and let T be a tree with maximum degree at most Δ. Then there exists a tree T' with $|V_T| = |V_{T'}|$, $diam(T') \leq 2\lceil \frac{diam(T)}{i} \rceil$ and $cg(T',T) \leq \frac{(\Delta-1)^i-1}{\Delta-2}$.*

Proof. Let v be a vertex of degree $\leq \Delta - 1$ in T (since T is a tree, such a vertex always exists). For each $x \in V_T$, $d(x,v) \leq diam(T)$. It is easy to observe that T can be extended to the complete $(\Delta-1)$-ary tree $L = R(\Delta-1, diam(T)+1)$ by successively adding vertices of degree one. Let S be the set of all added vertices to T. Let us consider the graph $G = F((\Delta-1)^i; \Delta-1, diam(T)+1)$. By Lemma 2.4, $diam(G) \leq 2\lceil \frac{diam(T)}{i} \rceil$. Moreover, by Lemma 2.5, there is an embedding (ϕ,ψ) of G in L with $cg_{(\phi,\psi)}(G,L) \leq \frac{(\Delta-1)^i-1}{\Delta-2}$. Let T' be defined as $T' = G - \{x \ : \ \phi(x) \in S\}$. In what follows we show that T' is indeed the required tree.

Let ϕ' be the restricted mapping ϕ to the graph T'. Obviously, ϕ' is an injection. Since ϕ' is a surjection as well, $|V_{T'}| = |V_T|$. Since ψ maps each edge xy to the unique $\phi(x) - \phi(y)$ path in L and since L and T are trees, the restricted mapping ψ to the graph T', say ψ', is well defined.

Hence (ϕ',ψ') is an embedding of T' in T with $cg_{(\phi',\psi')}(T',T) \leq cg_{(\phi,\psi)}(G,L)$ $\leq \frac{(\Delta-1)^i-1}{\Delta-2}$. Since G is a tree, to finish the proof it is sufficient to prove that T' is connected, since then $diam(T') \leq diam(G) \leq 2\lceil \frac{diam(T)}{i} \rceil$. Let x be the last vertex added to T in the construction of L. Since x is a leaf in L and by Remark 2.1, it follows that $\phi^{-1}(x)$ is a leaf in G as well. Thus after removing x and $\phi^{-1}(x)$ the resulting graphs will be connected. We can continue removing vertices from L and its preimages in G (in the opposite way than they were added to T) to obtain T and T', respectively. Since in each step the deleted vertex is a leaf of a subgraph of L, by Remark 2.1, its preimage is also a leaf of a subgraph of G, and thus T' is connected. $\square$

THEOREM 2.7. *For any graph H of maximum degree $\Delta \geq 3$ and any given $c \geq 1$,*

$$Hop_H(c) = O\left(\frac{diam(H)\log\Delta}{\log c}\right).$$

Proof. Let T be a breadth first search spanning tree of G. Then T has the maximum degree at most $\Delta \geq 3$ and $diam(T) \leq 2diam(H)$. Find i such that $\frac{(\Delta-1)^i-1}{\Delta-2} \leq c < \frac{(\Delta-1)^{i+1}-1}{\Delta-2}$. Now, by Lemma 2.6, there is a tree T' with $cg(T',T) \leq \frac{(\Delta-1)^i-1}{\Delta-2}$. Clearly $cg(T',H) \leq \frac{(\Delta-1)^i-1}{\Delta-2} \leq c$. Similarly, by Lemma 2.6,

$$\begin{aligned} Hop_H(c) \leq diam(T') \leq 2\left\lceil\frac{diam(T)}{i}\right\rceil &\leq 2\left\lceil\frac{2diam(H)\log(\Delta-1)}{\log(\Delta-2)c+1)-\log(\Delta-1)}\right\rceil \\ &= O\left(\frac{diam(H)\log\Delta}{\log c}\right). \qquad \square \end{aligned}$$

Theorem 2.7 together with Lemma 2.3 has two important consequences.

COROLLARY 2.8. *Let H be a graph of order n with $\Delta = O(1)$ and $diam(H) \leq O(\log n)$. Then $Hop_H(c) = \Theta(\frac{\log n}{\log c})$ for any c.*

Note that several standard networks belong to this class of graphs, e.g. mesh of trees, Butterfly, cube-connected-cycles, binary de Bruijn or shuffle-exchange graph.

COROLLARY 2.9. *Let H be a graph of order n with maximum degree $\Delta \geq 3$ satisfying $diam(H) \log \Delta = O(\log n)$. Then $Hop_H(c) = \Theta(\frac{\log n}{\log c})$ for $c \geq \Delta$.* If $c \geq \Delta$, then the above corollary gives optimal virtual path layouts for e.g. star, pancake and k-ary de Bruijn and Kautz graphs.

3. Paths and Meshes. Let P_n denote an n vertex path and $M_n = P_n \times P_n$ denote an $n \times n$ mesh. The next theorem improves virtual path layouts for paths and meshes of Kranakis et al. [6].

THEOREM 3.1. *Let $c \geq 1$. Then*
(*i*)

$$Hop_{P_n}(c) = \Theta\left(\frac{\log n}{\log c}\right), \text{ for } c \geq \log^{1+\varepsilon} n, \text{ for any fixed } \varepsilon > 0.$$

(*ii*)

$$Hop_{M_n}(c) = \Theta\left(\frac{\log n}{\log c}\right), \text{ for } c \geq 2.$$

Proof. Both lower bounds easily follow by applying Lemma 2.3. Consider the upper bounds. To prove (*i*), first find m and $k = 2^i$ such that

$$2^m - 1 \leq n < 2^{m+1} - 1 \text{ and } (k-1)(m+2)/2 + 1 \leq c < (2k-1)(m+2)/2 + 1.$$

Setting $G_1 = F(k; 2, m), G_2 = B_m$ and $G_3 = P_{2^m-1}$ in Lemma 2.2, using a result of Lengauer (based on the orthogonal projection of B_m on a horizontal line) [8]: $cg(B_m, P_{2^m-1}) \leq (m+2)/2$ and by Lemma 2.5 we get an embedding of $F(k; 2, m)$ into P_{2^m-1} with

$$cg(F(k; 2, m), P_{2^m-1}) \leq (k-1)(m+2)/2.$$

Now choose $n - 2^m + 1$ new vertices and join them to leaves of $F(k; 2, m)$ (at most two per each leaf) in such a way that the resulting graph, say T, has $m + 1$ levels. Note that this is always possible. Clearly, T has n vertices. It is easy to extend the above embedding of $F(k; 2, m)$ into P_{2^m-1} to an embedding of T into P_n with

$$cg(T, P_n) \leq cg(F(k; 2, m), P_{2^m-1}) + 1 \leq c.$$

Moreover, by Lemma 2.4

$$\begin{aligned} Hop_{P_n}(c) &\leq diam(T) \leq diam(F(k; 2, m)) + 2 = 2\left\lceil \frac{m-1}{i} \right\rceil + 2 \leq \frac{2m}{\log k} + 4 \\ &\leq \frac{2\log(n+1)}{\log \frac{c}{2\log n}} + 4 \leq \frac{2\log(n+1)}{\frac{\varepsilon}{1+\varepsilon}\log c - 1} + 4 = O\left(\frac{\log n}{\log c}\right), \end{aligned}$$

for $c \geq \log^{1+\varepsilon} n$.

For (ii) assume first the case $n = 2^m, m \geq 1$. Find $k = 2^i$ such that

$$2(k-1)+1 \leq \frac{c}{4} < 2(2k-1)+1.$$

Set $G_1 = F(k;2,2m), G_2 = B_{2m}$, and $G_3 = M_n$ in Lemma 2.2. Zienicke [11] showed that $cg(B_{2m}, M_n) = 2$. Now we add one new vertex and edge to $F(k;2,2m)$ resulting in a graph F on n vertices embeddable into M_n with $cg(F, M_n) \leq 2(k-1)+1 \leq c/4$. The diameter of F is increased by at most 1.

Second, consider an arbitrary n. Find m such that $2^m - 1 \leq n < 2^{m+1} - 1$. Embed the graph F into left lower submesh of M_n of size $2^m \times 2^m$. Similarly embed the graph F into other three "corner" submeshes of M_n. We get a virtual path layout for M_n with congestion at most $4cg(F, M_n) \leq c$ and the hop-number

$$\begin{aligned} Hop_{M_n}(c) &\leq 2diam(F) \leq 2(diam(F(k;2,2m))+1) \leq 2\left\lceil \frac{2m-1}{i} \right\rceil + 2 \\ &\leq 2\frac{2\log(n+1)}{\log(\frac{c+4}{16})} + 4 = O\left(\frac{\log n}{\log c}\right). \qquad \square \end{aligned}$$

Note that $Hop_{P_n}(c)$ for non-constant $c \leq \log n$ remains an open problem.

4. Hypercubes. The main result of this section is an optimal virtual path layout for hypercubes. The n-dimensional hypercube, denoted by Q_n, is defined, by means of Cartesian product of graphs, as $Q_0 = v$ (a single vertex) and $Q_n = Q_{n-1} \times P_2$. Edges of Q_n are divided into n groups in a natural way according to dimensions they belong to.

Let us define the tree T_1 to be a single vertex. Now, for $n \geq 2$ let the tree $T_n = S_{2k-2} * kT_{n-i}$, where $k = 2^i \leq n < 2^{i+1}$. Note that the number of vertices of T_n is $2^n - 1$.

LEMMA 4.1. *The diameter of T_n satisfies*

$$diam(T_n) < \frac{8n}{\lfloor \log n \rfloor} + 2\lfloor \log n \rfloor.$$

Proof. Let $depth(T_n)$ denote the depth of T_n. Then

$$depth(T_n) = depth(T_{n-i}) + 1,$$

where $i = \lfloor \log n \rfloor$. Solving this recurrence we get

$$\begin{aligned} depth(T_n) &\leq \left\lceil \frac{n-2^i+1}{i} \right\rceil + \left\lceil \frac{2^i - 2^{i-1}}{i-1} \right\rceil + \ldots + \left\lceil \frac{2^2-2^1}{1} \right\rceil \\ &< \sum_{j=1}^{i} \left\lceil \frac{2^j}{j} \right\rceil \leq \sum_{j=1}^{i} \frac{2^j}{j} + i \leq \frac{2^{i+2}}{i} + i \leq \frac{4n}{\lfloor \log n \rfloor} + \lfloor \log n \rfloor, \end{aligned}$$

where the estimation of the sum was done by a straightforward induction. Noting that $diam(T_n) \leq 2depth(T_n)$ we have the result. $\square$

Let G be a graph and let S (U) be a subset of E_G (V_G). We define the *graph induced by U under S* as the graph with the vertex set U and all edges $xy \in (U \times U) \cap S$.

Let S be the set of all edges in some $i \leq n$ dimensions of Q_n. Then the graph $Q_n - S$ consists of 2^i copies of Q_{n-i}, say $Q^1, Q^2, \ldots, Q^{2^i}$. Let v^1 be a vertex of Q^1.

Let $v^l \in Q^l$, $2 \leq l \leq 2^i$ be the corresponding vertices to v^1 i.e. $v^l, 2 \leq l \leq i$ is the copy of v^1 chosen in the natural way. Then the graph induced by $\{v^1, v^2, \ldots, v^{2^i}\}$ under S is an i-dimensional hypercube, say $Q_i(v^1)$. Observe that the hypercubes $Q_i(x)$ for all $x \in Q^1$ are vertex disjoint. Moreover, the mapping $\xi : Q^1 \to Q^l$, $l = 2, \ldots, 2^i$, given by $\xi(x) = y$, where $y \in Q^l \cap Q_i(x)$, is the isomorphism induced by $Q_i(v^1)$.

LEMMA 4.2. *For all* $n \geq 1$, $cg(T_n, Q_n) \leq 2$.

Proof. In fact we prove, by induction on n, the following stronger statement: There exists an embedding (ϕ, ψ) of T_n into Q_n with $cg(e) \leq 2$, for all edges $e \in Q_n$, satisfying the following two additional conditions. Let r be the root of T_n and $v \in V_{Q_n} \setminus \phi[V_{T_n}]$.

(i) There exists a $\phi(r) - v$ path P such that each edge of P has congestion at most 1 and just one neighbour of v is on P.

(ii) For all neighbours x of v in Q_n, $cg(vx) = 0$.

Let us call the vertex v a *free vertex*, the path P a *free path* and the embedding a *good embedding*. For $n = 1, 2, 3$ the statement is easy to observe. Thus we assume we have proved the statement for all integers up to $n - 1$ and we prove it for $n \geq 4$. By definition, $T_n = S_{2k-2} * kT_{n-i}$, where $k = 2^i \leq n < 2^{i+1}$. Let us delete all edges in some i dimensions of Q_n obtaining thereby k copies of Q_{n-i}, say $Q^1, \ldots, Q^k$ (see Fig. 2).

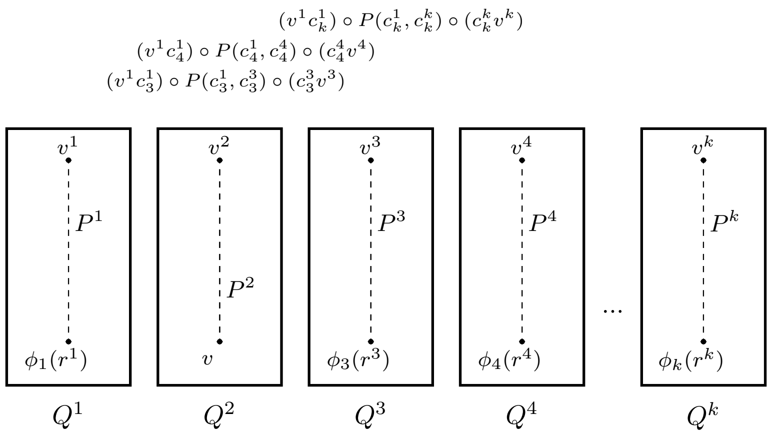

Fig. 2 The k subhypercubes Q_{n-i}

For the notational convenience let us denote some important vertices of T_n. By r we denote the root of T_n (this is also the root of S_{2k-2}). Further, by r^l we denote the leaf of S_{2k-2} which is the root of $F^l = T_{n-i}$ $(l = 1, 2, \ldots, k)$. Note that r^l are the vertices at which the subtrees T_{n-i} are amalgamated with S_{2k-2} in T_n. Finally, by x^l $(l = 3, 4, \ldots, k)$ we denote the $k - 2$ remaining leaves of S_{2k-2}.

Let v^1 be any vertex of Q^1. In what follows we describe for $l = 1, \ldots, k$ a good embedding of F^l into Q^l. By the induction hypothesis there is a good embedding (ϕ_1, ψ_1) of F^1 into Q^1. Since Q^1 is vertex transitive, we may assume that v^1 is the free vertex in Q^1. Moreover, there is a free $\phi_1(r^1) - v^1$ path, say P^1. By v^l we denote the vertex corresponding to v^1 in Q^l. It follows from the symmetry of Q_{n-i} that there is a good embedding (ϕ_2, ψ_2) of F^2 into Q^2 (obtained from the embedding of F^1 into Q^1) for which $\phi_2(r^2) = v^2$ and the free vertex in Q^2, say v, is the vertex corresponding to $\phi_1(r^1)$. Furthermore, there is a $v^2 - v$ free path P^2 in Q^2. Finally, for $l = 3, \ldots, k$, by the existence of the isomorphism induced by $Q_i(v^1)$, there is a good embedding (ϕ_l, ψ_l) of F^l into Q^l for which $\phi_l(r^l)$ is the vertex corresponding to $\phi_1(r^1)$ and v^l is the free vertex in Q^l. By the same argument, there is a $v^l - \phi_l(r^l)$

free path P^l in Q^l, $l = 3, \ldots, k$.

Since $Q^1 \cup Q^2 \cup \ldots \cup Q^k$ is a factor of Q_n and since $F^1 \cup F^2 \cup \ldots \cup F^k \cong kT_{n-i}$, the embeddings (ϕ_l, ψ_l), $l = 1, \ldots, k$, together induce an embedding (ϕ, ψ) of kT_{n-i} into $Q_n \setminus \{v, v^1, v^3, v^4, \ldots, v^k\}$. To obtain the required embedding we put $\phi(r) = v^1$ and $\phi(x^l) = v^l$ for $l = 3, 4, \ldots, k$. Finally, we have to embed the edges of S_{2k-2}. Since all the vertices of S_{2k-2} are already embedded, it is enough to describe the paths $v^1 - \phi(r^l)$, $l = 1, 2, \ldots, k$, and the paths $v^1 - v^l$, $l = 3, 4, \ldots, k$.

The path $v^1 - \phi(r^1) = P^1$ and the path $v^1 - \phi(r^2) = v^1v^2$. We construct the remaining paths using neighbours of the vertex v^1 in Q^1. Since the degree of Q_n is $n \geq k$ and since P_1 is a free path, there are $k-2$ neighbours of v^1 which lie neither on P^1 nor on v^1v^2, say $c_3^1, c_4^1, \ldots, c_k^1$. For each vertex c_j^1, $j = 3, 4, \ldots, k$, there is the hypercube $Q_i(c_j^1)$. Let c_j^l be the corresponding vertex for c_j^1 in Q^l. Since the mapping on corresponding vertices is an isomorphism, the vertex c_j^l is adjacent to the vertex v^l. Since $Q_i(c_j^1)$ $(j = 3, 4, \ldots, k)$ is connected, there is a $c_j^1 - c_j^l$ path, say $P(c_j^1, c_j^l)$, in $Q_i(c_j^1)$. Note that all $P(c_j^1, c_j^l)$ for $j = 3, 4, \ldots, k$ are vertex disjoint. Now, we are able to describe all remaining paths. For $j = 3, 4, \ldots, k$, the path $v^1 - v^j = (v^1c_j^1) \circ P(c_j^1, c_j^j) \circ (c_j^jv^j)$, and the path $v^1 - \phi(r^j) = (v^1c_j^1) \circ P(c_j^1, c_j^j) \circ (c_j^jv^j) \circ P^j$, where $\circ$ it the concatenation operation.

The embedding is now completely defined. It is easy, but time consuming exercise, to observe that $cg(e) \leq 2$, for each edge $e \in Q_n$. If we define $P = (v^1v^2) \circ P^2$, then since P_2 is a free path, P is a $\phi(r) - v$ free path as well. Moreover, no edge incident with v is used in the embedding, thus $cg(vx) = 0$ for all neighbours x of v and the embedding (ϕ, ψ) is good. □

THEOREM 4.3. *For the n-dimensional hypercube we have*

$$Hop_{Q_n}(c) = \begin{cases} \Theta\left(\frac{n}{\log n}\right) & \text{if } 2 \leq c \leq n, \\ \Theta\left(\frac{n}{\log c}\right) & \text{otherwise.} \end{cases}$$

Proof. The upper bound of the first case follows from Lemmas 4.1 and 4.2. The lower bound is implied by Lemma 2.3. The second case is proved in a similar way as the result for the mesh using the fact from [7] that the complete binary tree on $2^n - 1$ vertices can be embedded in the n-dimensional hypercube with congestion 1. □

In particular, the above result says that if the congestion c is smaller than the degree of the hypercube then the hop-number does not depend on c. It is an interesting open question whether this holds for other important networks of unbounded degree like k-ary de Bruijn and star graph. Another open problem is to find an optimal virtual paths layout for the n-vertex path if c is non-constant and less than or equal to $\log^{1+\varepsilon} n$, for $\varepsilon > 0$.

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