Bisection Widths of Transposition Graphs and their Applications

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Abstract

We prove lower and upper bounds on bisection widths of the transposition graphs. This class of graphs contains several frequently studied interconnection networks including star graphs and hypercubes. In particular, we prove that the bisection width of the complete transposition graph is of order $\Theta(n.n!)$ which solves the open problem (R) 3.356 of the Leighton's book [10] and determine nearly exact value of bisection width of the star graph. The results have applications to VLSI layouts, cutwidths and crossing numbers of transposition graphs. We also study bandwidths of these graphs.

1 Introduction

Several interconnection networks of parallel computers based on permutations have appeared recently, e.g. bubblesort and star graph, alternating group graphs..., see a survey paper of Lakshmivarahan et al. [8]. Leighton [10] introduced the $n-$dimensional complete transposition graph CT_n. It consists of $n!$ vertices, each of which corresponds to a permutation on n numbers. Two vertices are adjacent iff their corresponding permutations differ by a single transposition. Leighton suggested to study a computational power of a parallel computer with the complete transposition graph interconnection network. Especially, he asked what the bisection width of the complete transposition graph is (problem (R) 3.356). The bisection width is the size of the smallest edge cut of a graph which divides it into two equal parts. This graph invariant is a fundamental concept in the theory of interconnection networks of parallel computers. It influences both algorithmic design and lower bound techniques [9, 12, 13]. We study the problem in a more general framework. For a given graph G with vertices labeled by $1, 2, \ldots, n$ its transposition graph $T(G)$ is a Cayley graph on a permutation group with generators-transpositions $\{(ij)\}$ whenever ij is an edge in G (see. [8]). Transposition graphs, when G is a tree, a cycle or a complete graph, were studied by Akers et al. [1] and Jerrum [6]. Kompel'makher and Liskovets [7] and Rosa et al. [15] proved that if G is connected, then the corresponding transposition graph is hamiltonian. The main contribution of our paper is a unified look on several

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basic parameters of transposition graphs. We give lower and upper bounds on bisection widths of transposition graphs. In particular, we prove that the bisection width of CT_n is of order $\Theta(n.n!)$ and the bisection width of the star graph S_n is $n!/4 + O(n-1)!$. We apply the results to optimal VLSI layouts, cutwidths and crossing numbers of transposition graphs. We also show an upper bound for the bandwidth of transposition graphs, which can not be improved in general.

2 Preliminaries

Throughout, $G = (V_G, E_G)$ will be a graph with $V_G = \{1, 2, \ldots, n\}$, where $n \geq 3$. Let Σ_n be the full symmetric group acting on the set $\{1, 2, \ldots, n\}$ and X to be the set of transpositions (permutations (i, j) - that is: i maps to j, j maps to i and everything else is fixed) which correspond to edges of G, i.e., $X = \{(ij) \in \Sigma_n \mid ij \in E_G\}$.

Let G be a graph and let Γ be the group which is generated by the set X. Then the *transposition graph* $T(G)$ of G is the Cayley graph $C(\Gamma, X)$ of the group Γ for the generating set X. In any case we may define the Cayley graph $C(\Gamma, X)$ in the obvious way: The vertex set of $C(\Gamma, X)$ is the group Γ, and two vertices (=permutations) π and σ are adjacent iff $\pi\tau = \sigma$ for some $\tau \in X$. Note that if $\pi\tau = \sigma$, then also $\pi = \sigma\tau$ since $\tau^2 = id$, where id denotes the identity permutation. It is worth noting that the composition $\pi\sigma$ will be understood "from right to left", i.e., $\pi\sigma(i) = \pi(\sigma(i))$.

For example, if $G = K_{1,n-1}$, $G = K_n$, $G = P_n$ - the n-vertex path, or $G = C_n$ - the n-vertex cycle, then $T(G)$ is well-known star graph [1] S_n, complete transposition graph CT_n, bubblesort graph and modified bubblesort graph, respectively. Moreover, if G is a union of m distinct edges, then $T(G) = Q_m$ - the m-dimensional hypercube graph [1, 8].

By definition, two vertices (permutations) of $T(G)$, π and σ are adjacent if and only if $\pi\tau = \sigma$ for some $\tau \in X$. In other words, the two permutations differ by a single transposition. Observe that if π and σ are adjacent in $T(G)$, then there is exactly one transposition $\tau \in X$ for which $\pi\tau = \sigma$. We say that the edge $x = \pi\sigma \in E_{T(G)}$ has *colour* τ and write $col(x) = \tau$. For a permutation π, expressed in cycle form, let $c(\pi)$ denote the number of cycles in π. Similarly, let $c_1(\pi)$ denote the number of cycles of π of length one.

Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree in a graph G, respectively. Let $d_G(u, v)$ denote the distance between vertices u and v in G. Let $bw(G)$ denote the bisection width of a graph G, i.e. the minimal number of edges the removal of which divides G in two disconnected parts having $\leq \lceil |V_G|/2 \rceil$ vertices each. The crossing number $cr(G)$ of a graph G is defined as the least number of crossings of its edges when G is drawn in a plane.

We use the standard model for laying out VLSI circuits [11, 20]. The circuit is viewed as a graph G in which vertices correspond to processing elements and edges to wires. The graph is then embedded in a two-dimensional grid subject to the following assumptions:

- A vertex of degree d occupies a square of side d. The squares corresponding to distinct vertices do not overlap.
- Edges have unit width and are routed along grid lines with the restriction that no two edges overlap except possibly when crossing perpendicular to each other or

[1] Note that the "star graph" here is not the graph $K_{1,n-1}$ for which the term is used more commonly but the transposition graph of $K_{1,n-1}$

when bending. Also, an edge can not be routed over a node which does not connect. The area of a two-dimensional layout of G is defined as the area of the "boundary rectangle" and it equals the product of the number of vertical tracks and the number of horizontal tracks that contain a node or a wire segment of the graph. The area is denoted by $area(G)$.

The cutwidth and the bandwidth are another important parameters of interconnection networks studied in connection with VLSI layouts [2, 16]. For a labeling $\psi : V_G \rightarrow \{1, 2, ..., n\}$ of a graph $G = (V_G, E_G)$ we define

$$cw_\psi(G) = \max_{1 \le i < n} |\{xy \in E_G \mid \psi(x) \le i < \psi(y)\}|$$

$$band_\psi(G) = \max_{xy \in E_G} \{|\psi(x) - \psi(y)|\}.$$

The cutwidth of G, denoted by $cw(G)$ and the bandwidth of G, denoted by $band(G)$, is the minimum value $cw_\psi(G)$ and $band_\psi(G)$, respectively, over all labelings ψ of G.

3 Transposition Graphs and their Properties

In this section we introduce some basic properties of transposition graphs and prove several useful lemmas.

Recall, G denotes a graph on n vertices $1, 2, \ldots, n$ and X the set of all transpositions corresponding to edges of G. If G is connected, then X generates Σ_n. Similarly, it is easy to observe that if G consists of k components $G_1, G_2, \ldots, G_k$, with $|G_i| = n_i$ $(i = 1, 2, \ldots, k)$, then X generates $\Sigma_{n_1} \times \Sigma_{n_2} \times \ldots \times \Sigma_{n_k}$. Thus we have the following consequence. If G is a graph with k connected components $G_1, G_2, \ldots, G_k$ such that $|V_{G_i}| = n_i (i = 1, 2, ..., k)$, then $|V_{T(G)}| = \prod_{i=1}^k n_i!$ and $\delta(T(G)) = \Delta(T(G)) = |E_G|$.

Let $Aut(G)$ be the group of all automorphisms of a graph G. If M is a subset of E_G, put $X_M = \{(i, j) \in \Sigma_n \mid ij \in M\}$. The following lemma has been proved in [15].

Lemma 3.1 *Let M be an edge orbit under the action of $Aut(G)$ on E_G. Then for any two edges x and y of $T(G)$ such that $col(x), col(y) \in X_M$ there is an automorphism of $T(G)$ mapping x onto y.*

Lemma 3.2 *Let G be a graph with $k \ge 1$ connected components $G_1, G_2, \ldots, G_k$ such that $|V_{G_i}| = n_i, n_i > 1 \quad (i = 1, 2, \ldots, k)$. Then*

$$\sum_{\pi \in T(G)} c(\pi) = \prod_{i=1}^{k} n_i! \sum_{i=1}^{k} H_{n_i}$$

and

$$\sum_{\pi \in T(G)} c_1(\pi) \le \frac{ek}{2} \prod_{i=1}^{k} n_i!,$$

where $e = \lim_{l \to \infty}(1 + \frac{1}{l})^l$ and $H_m = \sum_{l=1}^{m} \frac{1}{l}$.

Proof. We start with the first statement. Assume G is connected, i.e. $k = 1$. Then $V_{T(G)} = \Sigma_n$. It is known [14], that the average number of cycles in a random permutation equals H_n. Hence

$$\sum_{\pi \in V_{T(G)}} c(\pi) = n! H_n.$$

Let G have $k > 1$ connected components $G_1, G_2, \ldots, G_k$. Since these components are vertex disjoint, any permutation $\pi \in T(G)$ can be expressed as a composition $\pi = \pi_1 \circ \pi_2 \circ \ldots \circ \pi_k$, where $\pi_i \in T(G_i)$ $(i = 1, 2, \ldots, k)$. Then

$$\begin{aligned}\sum_{\pi \in V_{T(G)}} c(\pi) &= \sum_{\pi \in V_{T(G)}} \sum_{i=1}^{k} c(\pi_i) = \sum_{i=1}^{k} \sum_{\pi \in V_{T(G)}} c(\pi_i) = \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} n_j! \sum_{\pi' \in V_{G_i}} c(\pi') \\ &= \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} n_j! n_i! H_{n_i} = \prod_{i=1}^{k} n_i! \sum_{i=1}^{k} H_{n_i}.\end{aligned}$$

The second statement. Assume again that G is connected. Let D_m denote the number of permutations of m elements without cycles of length 1. It is known from [14] that $D_m = m! \sum_{i=0}^{m} (-1)^i / i! \leq m!/2$. It is easy to see that

$$\sum_{\pi \in V_{T(G)}} c_1(\pi) = \sum_{l=0}^{n} l \binom{n}{l} D_{n-l} \leq \sum_{l=0}^{n} l \binom{n}{l} \frac{(n-l)!}{2} = \frac{n!}{2} \sum_{l=1}^{n} \frac{1}{(l-1)!} < \frac{en!}{2}.$$

If G has $k > 1$ connected components, we generalize the previous formula similarly to the first one. □

Lemma 3.3 *For any graph G and any $\pi \in V_{T(G)}$ it holds that*

$$d_{T(G)}(\pi, id) \leq 2 \sum_{i=1}^{n} d_G(i, \pi(i)) - n - c(\pi) + 2c_1(\pi).$$

Proof. The distance between $\pi \in V_{T(G)}$ and id in $T(G)$ can be viewed as the minimum number of (not necessary distinct) transpositions from X whose composition with π results in id. We may assume π is expressed in the cycle representation

$$\pi = (a_{1,1}\, a_{1,2} \ldots a_{1,m_1})(a_{2,1},\, a_{2,2} \ldots a_{2,m_2}) \ldots (a_{l,1}\, a_{l,2} \ldots a_{l,m_l}),$$

where $l \geq 1$ and $m_j \geq 1$ $(j = 1, 2, \ldots, l)$. It is a matter of routine to observe that

$$[\ldots[[(a_{j,1}\, a_{j,2} \ldots a_{j,m_j}) \circ (a_{j,m_j}\, a_{j,m_j-1})] \circ (a_{j,m_j-1}\, a_{j,m_j-2})] \circ \ldots] \circ (a_{j,2}\, a_{j,1}) = id, \tag{1}$$

for $j = 1, 2, \ldots, l$. Thus one can see that the set of transpositions $\{(i\, \pi(i)) \mid i = 1, 2, \ldots, n\}$ contains a subset such that a composition of π with these transpositions results in id. Not all of these transpositions must belong to X. If, for example, $(a_k\, \pi(a_k)) \notin X$, then $a_k \pi(a_k) \notin E_G$. Since π is generated by transpositions from X and since X consists of transpositions corresponding to edges of G, we have that there is a $a_k - \pi(a_k)$ path in G, say $a_k x_1 x_2 \ldots x_t \pi(a_k)$ is a shortest one. The following composition shows how the transposition $(a_k\, \pi(a_k))$ can be replaced by $2d_G(a_k, \pi(a_k)) - 1$ transpositions from X; where $t = d_G(a_k, \pi(a_k)) - 1$.

$$[\ldots[(a_k\, x_1) \circ (x_1\, x_2)] \circ \ldots] \circ (x_{t-1}\, x_t)] \circ (x_t\, \pi(a_k))] \circ (x_t\, x_{t-1})] \circ \ldots] \circ (x_1\, a_k) = (a_k\, \pi(a_k))$$

Hence,

$$d_{T(G)}(\pi, id) \leq \sum_{i \neq \pi(i)} (2d_G(i, \pi(i)) - 1) = \sum_{i=1}^{n} (2d_G(i, \pi(i)) - 1) + c_1(\pi).$$

Since for any cycle $(a_{j,1}\, a_{j,2} \ldots a_{j,m_j})$ of π, with $m_j \geq 2$, the transposition $(a_{j,m_j}\, a_{j,1}) = (a_{j,m_j}\ \pi(a_{j,m_j}))$ is not in (1), we can decrease the previous estimation by $c(\pi) - c_1(\pi)$. Thus,

$$d_{T(G)}(\pi, id) \leq 2\sum_{i=1}^{n} d_G(i, \pi(i)) - n - c(\pi) + 2c_1(\pi). \quad \square$$

If G is a forest, then a better bound holds. The following lemma was originally proved for trees in [1]. A direct inspection of the proof shows that it is valid for forests too.

Lemma 3.4 *For any forest G it holds that*

$$d_{T(G)}(\pi, id) \leq \sum_{i=1}^{n} d_G(i, \pi(i)) - n + c(\pi).$$

4 Bisections

4.1 Lower Bounds

Shahrokhi and Székely [17] generalized Leighton's lower bound method [9] for estimating bisection widths in the following formula. The notation (i, j) will refer to an ordered pair of vertices i and j.

Theorem 4.1 *For each graph G let E_G^1 denote its smallest edge orbit under the action of automorphism group of G. Then*

$$bw(G) \geq \frac{|E_G^1||V_G|^2}{2\sum_{(i,j)\in V_G} d_G(i,j)}. \tag{2}$$

An application of Theorem 4.1 to transposition graphs gives

Theorem 4.2 *Let G be a graph with $k \geq 1$ connected components $G_1, G_2, \ldots, G_k$ such that $|V_{G_i}| = n_i, n_i > 1$, for $i = 1, 2, ..., k$. Then*

$$bw(T(G)) \geq \frac{|E_G^1| \prod_{i=1}^{k} n_i!}{4(\sum_{i=1}^{k} \frac{2}{n_i} \sum_{(x,y)\in V_{G_i}} d_G(x, y) - n - \sum_{i=1}^{k} H_{n_i} + ek)}.$$

Proof. Since X contains only involutions (i.e. if the image of a is b, then the image of b is a), the set of all edges of $T(G)$ of a given colour constitutes a 1-factor in $T(G)$. It follows that $T(G)$ is a union of $|X|$ 1-factors. This also implies that every vertex is incident with an edge of an arbitrary prescribed colour. Thus, for each edge (= transposition τ) of G there are exactly $\frac{1}{2}\prod_{i=1}^{k} n_i!$ edges of the same colour τ in $T(G)$.

Let M be an edge orbit under action of $Aut(G)$ on E_G. Let $X_M = \{(k\, l) \in \Sigma_n \mid kl \in M\}$ and let $M' = \{e \in E_{T(G)} \mid \exists\, \tau \in X_M, col(e) = \tau\}$, called *lift* of M. By Lemma 3.1, it follows that M' is a subset of an edge orbit under the action of $Aut(T(G))$ on $E_{T(G)}$. Moreover, it follows by Lemma 3.1 that each edge orbit under the action of $Aut(T(G))$

on $E_{T(G)}$ contains a lift of some edge orbit under action of $Aut(G)$ on E_G. By the above observations we have $|M'| = \frac{1}{2}|X_M| \prod_{i=1}^{k} n_i!$. Thus $|E^1_{T(G)}| \geq \frac{1}{2}|E^1_G| \prod_{i=1}^{k} n_i!$.

Now it is sufficient to evaluate the denominator in the formula (2) applied to $(T(G))$.

$$2 \sum_{(a,b)\in V_{T(G)}} d_{T(G)}(a,b) \leq 2|V_{T(G)}| \sum_{\pi\in V_{T(G)}} d_{T(G)}(\pi, id)$$

$$\leq 2|V_{T(G)}| \sum_{\pi\in V_{T(G)}} (2\sum_{l=1}^{n} d_G(l,\pi(l)) - |V_G| - c(\pi) + 2c_1(\pi))$$

$$= 2|V_{T(G)}|(2\sum_{l=1}^{n} \sum_{\pi\in V_{T(G)}} d_G(l,\pi(l)) - |V_{T(G)}||V_G| - \sum_{\pi\in V_{T(G)}} (c(\pi) - 2c_1(\pi)))$$

$$\leq |V_{T(G)}|(2\sum_{i=1}^{k} \sum_{x\in V_{G_i}} \sum_{\pi\in V_{T(G)}} d_G(x,\pi(x)) - |V_{T(G)}||V_G| - \prod_{i=1}^{k} n_i!(\sum_{i=1}^{k} H_{n_i} - ek))$$

$$= 2|V_{T(G)}|(2\sum_{i=1}^{k} \sum_{x\in V_{G_i}} \sum_{y\in V_{G_i}} \sum_{\substack{\pi\in V_{T(G)}\\ \pi(x)=y}} d_G(x,y) - |V_{T(G)}||V_G| - \prod_{i=1}^{k} n_i!(\sum_{i=1}^{k} H_{n_i} - ek))$$

$$= 2|V_{T(G)}|(2\sum_{i=1}^{k} \sum_{x\in V_{G_i}} \sum_{y\in V_{G_i}} \frac{\prod_{j=1}^{k} n_j!}{n_i} d_G(x,y) - |V_{T(G)}||V_G| - \prod_{i=1}^{k} n_i!(\sum_{i=1}^{k} H_{n_i} - ek))$$

$$= 2|V_{T(G)}|^2(2\sum_{i=1}^{k} \frac{1}{n_i} \sum_{(x,y)\in V_{G_i}} d_G(x,y) - |V_G| - \sum_{i=1}^{k} H_{n_i} + ek). \quad \square$$

Corollary 4.1 *If G is an edge symmetric graph with $k \geq 1$ connected components $G_1, G_2, \ldots, G_k$, then*

$$bw(T(G)) \geq \frac{|E_G|(\frac{n}{k}!)^k}{4(\frac{2k^2}{n}\sum_{(x,y)\in V_{G_i}} d_G(x,y) - n - k(\ln\frac{n}{k} - e))}.$$

Proof. The graphs G_i are mutually isomorphic and $n_i = n/k$. Substituting this into Theorem 4.2 and noting that $H_m > \ln m$ we get the claimed result. $\square$

Corollary 4.2 *If G is edge symmetric and connected then*

$$bw(T(G)) \geq \frac{|E_G| n!}{4(\frac{2}{n}\sum_{(x,y)\in V_G} d_G(x,y) - n)}.$$

The complete transposition graph CT_n is the transposition graph of K_n.

Corollary 4.3 *It holds that*

$$bw(CT_n) \geq \frac{n.n!}{8}.$$

If G is a forest then using Lemmas 3.2 and 3.4 and the same approach as in the proof of Theorem 4.2 one can easily show

Theorem 4.3 *Let G be a forest with $k \geq 1$ connected components $G_1, G_2, \ldots, G_k$ such that $|V_{G_i}| = n_i, n_i > 1$, for $i = 1, 2, \ldots, k$. Then*

$$bw(T(G)) \geq \frac{|E_G^1| \prod_{i=1}^k n_i!}{4(\sum_{i=1}^k \frac{1}{n_i} \sum_{(x,y)\in V_{G_i}} d_G(x,y) - n + \sum_{i=1}^k H_{n_i})}.$$

Corollary 4.4 *If G is an edge symmetric forest with $k \geq 1$ connected components $G_1, G_2, \ldots, G_k$, then*

$$bw(T(G)) \geq \frac{|E_G|((\frac{n}{k})!)^k}{4(\frac{k^2}{n} \sum_{(x,y)\in V_{G_i}} d_G(x,y) - n + kH_{\frac{n}{k}})}.$$

According to [8] the $m-$dimensional hypercube is the transposition graph of the graph consisting of m independent edges. Setting $n = 2m, k = m$ in Corollary 4.4 we get the well known optimal lower bound.

Corollary 4.5 *It holds that*

$$bw(Q_m) \geq 2^{m-1}.$$

The star graph S_n is the transposition graph of $K_{1,n-1}$ [1]. Setting $k = 1$ and $G = K_{1,n-1}$ in Corollary 4.4 we get

Corollary 4.6 *It holds that*

$$bw(S_n) \geq \frac{n!}{4(1 - \frac{2}{n} - \frac{1}{n-1} + \frac{H_n}{(n-1)})} = \frac{n!}{4}\,(1 - o(1))\,.$$

4.2 Upper Bounds

In this subsection we prove a general upper bound for bisection widths of transposition graphs and apply it to hypercubes, complete transposition graphs and star graphs.

Theorem 4.4 *Let G be a graph with $k \geq 1$ connected components G_1, G_2, …, G_k, such that $|V_{G_i}| = n_i, n_i > 1$, for $i = 1, 2, \ldots, k$. Let x_i be a vertex with minimum degree in G_i. Denote $\sigma_{n_i} = n_i \pmod 2$. Then*

$$bw(T(G)) \leq \min_i \left\{ \left(\left\lfloor \frac{n_i}{2} \right\rfloor^2 \frac{\delta(G_i)}{n_i(n_i-1)} + \sigma_{n_i} \frac{n_i - 1}{4} \frac{\delta(G_i - x_i) + 2\delta(G_i)}{n_i(n_i-2)} \right) \prod_{j=1}^k n_j! \right\}.$$

Proof. Consider an arbitrary component G_i and assume (without any lose of generality) that $V_{G_i} = \{1, 2, \ldots, n_i\}$. Let $A_1(A_2)$ denote the set of all vertices (=permutations) π in $T(G)$ such that $1 \leq \pi(x_i) \leq \lfloor \frac{n_i}{2} \rfloor$ ($\lceil \frac{n_i}{2} \rceil + 1 \leq \pi(x_i) \leq n_i$). Let A_3 denote the set of all vertices (=permutations) π in $T(G)$ such that $\pi(x_i) = \lceil \frac{n_i}{2} \rceil$. Observe that since G_i is a component of G it holds for any permutation π in $T(G)$ that $\pi(x_i) \leq n_i$. Hence A_1, A_2 (A_1, A_2, A_3) is a partition of $V_{T(G)}$ if n_i is even (odd). Let x_l be a vertex of minimum positive degree in $G_i - x_i$. Let $A_{31}(A_{32})$ denote the set of all permutations π in A_3 such that $1 \leq \pi(x_l) \leq \lfloor \frac{n_i}{2} \rfloor$ ($\lceil \frac{n_i}{2} \rceil + 1 \leq \pi(x_l) \leq n_i$). Again, A_{31}, A_{32} is a partition of A_3.

Obviously, $|A_1| = |A_2|$ and $|A_{31}| = |A_{32}|$. Deleting all edges joining A_1 with A_2 if n_i is even, and all edges joining A_1 with A_2, A_{31} with A_{32}, A_1 with A_{32}, and A_2 with A_{31} if n_i is odd, results in an edge cut which divides $T(G)$ into two parts having at most $\lceil |V_{T(G)}|/2 \rceil$ vertices each. The cardinality of the cut is at most

$$\left\lfloor \frac{n_i}{2} \right\rfloor^2 (n_i-2)!\delta(G_i) \prod_{\substack{j=1\\j\neq i}}^{k} n_j! + \sigma_{n_i}(n_i-3)! \prod_{\substack{j=1\\j\neq i}}^{k} n_j! \left(\left\lfloor \frac{n_i-1}{2} \right\rfloor^2 \delta(G_i - x_i) + 2\left\lfloor \frac{n_i}{2} \right\rfloor^2 \delta(G_i) \right).$$

Indeed, the first term denotes the number of edges joining A_1 with A_2. If $\pi \in A_1$, $\sigma \in A_2$ and $\pi\sigma \in E_{T(G)}$, then since $\pi(x_i) \neq \sigma(x_i)$ there must exist $y_i \in V_{G_i}$ such that $\pi(y_i) = \sigma(x_i)$, $\pi(x_i) = \sigma(y_i)$ and for all $x \neq x_i, y_i$ $\pi(x) = \sigma(x)$. We have x_i fixed and for y_i there are at most $\delta(G_i)$ possibilities (since y_i must be adjacent to x_i). Similarly, there are at most $\lfloor \frac{n_i}{2} \rfloor$ possibilities for $\pi(x_i)$ and $\sigma(x_i)$, respectively. It can be observed that under these conditions there are $(n_i-2)! \prod_{\substack{j=1\\j\neq i}}^{k} n_j!$ possibilities to complete π and σ.

The second term contributes to the expression iff n_i is odd and can be explained similarly. Hence the cardinality of the cut is at most

$$\left\lfloor \frac{n_i}{2} \right\rfloor^2 (n_i-2)!\delta(G_i) \prod_{\substack{j=1\\j\neq i}}^{k} n_j! + \sigma_{n_i}(n_i-3)! \frac{(n_i-1)^2}{4} \prod_{\substack{j=1\\j\neq i}}^{k} n_j! \left(\delta(G_i - x_i) + 2\delta(G_i)\right),$$

which immediately implies the desired result. □

Corollary 4.7 *Let G be an edge symmetric graph with $k \geq 1$ connected components G_1, G_2, ..., G_k. Let x_i be a vertex with minimum degree in G_i. Denote $\sigma_{n_i} = n_i \pmod 2$. Then*

$$bw(T(G)) \leq \left(\left\lfloor \frac{n}{2k} \right\rfloor^2 \frac{\delta(G_i)}{\frac{n}{k}(\frac{n}{k}-1)} + \sigma_{\frac{n}{k}} \frac{\frac{n}{k}-1}{4} \frac{\delta(G_i - x_i) + 2\delta(G_i)}{\frac{n}{k}(\frac{n}{k}-2)} \right) \left(\left(\frac{n}{k}\right)! \right)^k.$$

This upper bound gives the well known bisection width of the hypercube graph.

Corollary 4.8 *For the m-dimensional hypercube it holds that*

$$bw(Q_m) \leq 2^{m-1}.$$

Corollary 4.9 *Assume G is edge symmetric and connected. Let x be an arbitrary vertex of G. Then*

$$bw(T(G)) \leq \left(\left\lfloor \frac{n}{2} \right\rfloor^2 \frac{\delta(G)}{n(n-1)} + \sigma_n \frac{n-1}{4} \frac{\delta(G-x) + 2\delta(G)}{n(n-2)} \right) n!.$$

For the complete transposition graph we get an upper bound which differs only by a multiplicative factor of 2 from the lower bound.

Corollary 4.10 *For $n \geq 3$, the bisection width of the complete transposition graph satisfies*

$$bw(CT_n) \leq \frac{(n+2)n!}{4}.$$

For the star graph, we get an upper bound which is exact up to the second order term.

Corollary 4.11 *For $n \geq 3$, the bisection width of the star graph satisfies*

$$bw(S_n) \leq \frac{n!}{4}\left(1 + \frac{2}{n-2}\right).$$

4.3 Applications

In this section we describe a general approach how to find area-efficient VLSI layouts of transposition graphs. As a special case, we get an optimal layout for the complete transposition graph. This result will also imply an optimal bound for the crossing number of this graph.

Let $H_n = G$ and let v_{H_n} be an arbitrary vertex of H_n belonging to a component F_{H_n} of H_n. Consider all permutations π in $T(H_n)$ such that $\pi(v_{H_n}) = m$, where $m \in F_{H_n}$. The graph induced by these permutations is isomorphic to $T(H_{n-1}) = T(H_n - v_{H_n})$. Hence, if m ranges over the set F_{H_n} then we partition $T(H_n)$ into $|F_{H_n}|$ distinct graphs isomorphic to $T(H_{n-1})$. Let w_n, h_n be the width and the height of the layout of the graph $T(H_n)$.

First suppose that $H_n - v_{H_n}$ has a connected component of at least 2 vertices. Place $|F_{H_n}|$ layouts of $T(H_{n-1})$ on a line with widths perpendicular to the line. Now we route the $|V_{T(H_n)}||E_{H_n}| - |F_{H_n}||V_{T(H_{n-1})}||E_{H_{n-1}}|/2$ remaining edges in the following way: Consider an edge that connects two vertices (squares) from different copies of $T(H_{n-1})$. We open two vertical tracks close to the right-hand sides of the squares. Then we add one new track above the temporary layout. See Fig. 1.

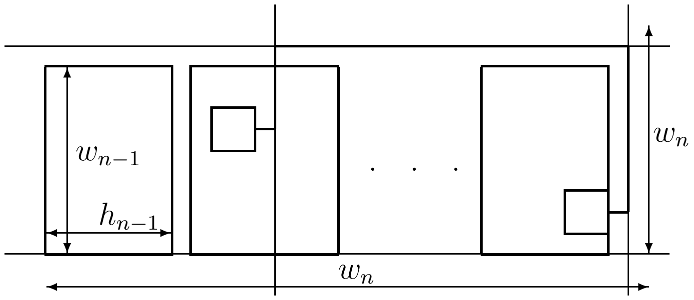

Fig. 1: The recursive layout of transposition graphs

We get a layout of $T(H_n)$ of the width and the height

$$w_n = |F_{H_n}|h_{n-1} + |V_{T(H_n)}||E_{H_n}| - |F_{H_n}||V_{T(H_{n-1})}||E_{H_{n-1}}|, \tag{3}$$

$$h_n = w_{n-1} + \frac{|V_{T(H_n)}||E_{H_n}|}{2} - \frac{|F_{H_n}||V_{T(H_{n-1})}||E_{H_{n-1}}|}{2}. \tag{4}$$

Second, suppose that H_{n-1} is a graph of isolated vertices. Then $T(H_{n-1})$ is a graph isomorphic to $(|F_{H_n}|-1)!$ isolated vertices. Put all vertices of $T(H_{n-1})$ on a line provided that each vertex has a square shape of the side $|E_{H_n}| = \Delta(T(H_n))$. Then

$$w_{n-1} = (|F_{H_n}| - 1)!|E_{H_n}|.$$

$$h_{n-1} = |E_{H_n}|.$$

Place $|F_{H_n}|$ layouts of $T(H_{n-1})$ on a line with widths perpendicular to the line. Now we route the $|V_{T(H_n)}||E_{F_n}|/2$ remaining edges similarly as above and get a layout of $T(H_n)$ of the width and the height

$$w_n = |F_{H_n}||E_{H_n}| + |V_{T(H_n)}||E_{F_n}|,$$

$$h_n = (|F_{H_n}| - 1)!|E_{H_n}| + |V_{T(H_n)}||E_{F_n}|/2.$$

The solution of the above recurrence relations heavily depends on the graph $H_n = G$ and a choice of the vertex v_{H_n}. If $G = K_{1,n-1}$ and v_{H_n} is a vertex of degree 1 then the solution of (3),(4) gives area optimal VLSI layout of the star graph with $area(S_n) = \Theta((n!)^2)$. Now we apply the method for the complete transposition graph.

Theorem 4.5 *The area of the optimal VLSI layout of the complete transposition graph satisfies*

$$area(CT_n) = \Theta((n.n!)^2).$$

Proof. The lower bound follows from Corollary 4.3 and a Thompson's result: $area(G) = \Omega(bw^2(G))$, [20].

In case $G = K_n$, the above approach gives $H_j = K_j$, for $j = 1, 2, ..., n$ and the equations (3) and (4) turn to

$$w_j = jh_{j-1} + j!(j-1),$$

$$h_j = w_{j-1} + \frac{j!(j-1)}{2},$$

for $j = 2, 3, ..., n$, with initial conditions

$$w_1 = h_1 = |E_{K_n}| = \frac{n(n-1)}{2}.$$

It is easy to verify that the solution of these relations satisfies

$$w_n = O(n.n!), h_n = O(n.n!). \quad \square$$

The cutwidth is a parameter that reflects the quality of linear VLSI layouts, i.e. when the components are placed on a line and wires are placed above the line using vertical and horizontal tracks. The minimal number of horizontal tracks over all placements equals the cutwidth of the underlying graph.

Theorem 4.6 *Cutwidths of the complete transposition and star graph satisfies:*

$$cw(CT_n) = \Theta(n.n!) \text{ and } cw(S_n) = \Theta(n!).$$

Proof: The lower bound follows from the fact that the cutwidth is bounded below by the bisection width. Consider the same notation as for the VLSI layout. We can easily construct a linear layout ψ of a graph $G = H_n$ by placing the vertices of the graphs H_{n-1}'s on a line and proceeding recursively. Then

$$cw_\psi(H_n) \leq cw_\psi(H_{n-1}) + \frac{|V_{T_{H_n}}||E_{H_n}|}{2} - \frac{|F_{H_n}||V_{T(H_{n-1})}||E_{H_{n-1}}|}{2}.$$

In the case of transposition and star graphs we easily verify that the solutions are $\Theta(n.n!)$ and $\Theta(n!)$, respectively. $\square$

Crossing numbers have been widely studied in graph theory but very little is known even on the approximative values of $cr(G)$ for explicitly given graphs [3, 18]. The importance of crossing numbers of graphs in VLSI complexity theory was pointed out by Leighton [9]. As a direct consequence of the above result we get an optimal upper bound on the crossing number of the transposition graph.

Theorem 4.7 *The crossing number of the complete transposition graph satisfies*

$$cr(CT_n) = \Theta((n.n!)^2).$$

Proof. The upper bound follows from a simple observation that $cr(G) \leq area(G)$. The lower bound follows from a formula of Shahrokhi and Székely [17]:

$$cr(G) = \Omega\left(\frac{|V_G|^4 |E_G^1|^2}{(\sum_{(i,j)\in V_G} d_G(i,j))^2} - \frac{|V_G|\Delta^2(G)}{2}\right),$$

where $|E_G^1|$ and $\sum_{(i,j)\in V_G} d_G(i,j)$ have the same meaning as in Theorem 4.1. $\square$

5 Bandwidths of Transposition Graphs

In this section we prove a general upper bound on the bandwidth of transposition graphs. The bound can not be improved in general.

Theorem 5.1 *Let G be a graph with $k \geq 1$ connected components $G_1, G_2, \ldots, G_k$ such that $|V_{G_i}| = n_i, n_i > 1$, for $i = 1, 2, \ldots, k$. Then*

$$band(G) \leq 2 \max_{k \leq l \leq n} \sum_{\substack{1 \leq x_i \leq n_i, 1 \leq i \leq k \\ x_1 + x_2 + \ldots + x_k = l}} \prod_{i=1}^{k} s(n_i, x_i),$$

where $s(n_i, x_i)$ is the sign-free Stirling number of the first kind.

Proof. Partition the vertices (=permutations) of $T(G)$ into n groups such that the l-th group, denoted by D_l contains all permutations having exactly l cycles. For all $l = k, k+1, ..., n$ consider bijections

$$f_k : D_k \to \{1, 2, ..., |D_k|\},$$

$$f_l : D_l \to \{1 + \sum_{i=k}^{l-1} |D_i|, 2 + \sum_{i=k}^{l-1} |D_i|, ..., |D_l| + \sum_{i=k}^{l-1} |D_i|\},$$

for $l > k$. These bijections define a labeling

$$f : V_{T(G)} \to \{1, 2, ..., |V_{T(G)}|\}.$$

Consider an edge $uv \in E_{T(G)}$. We claim that if $u \in D_l$ then v belongs either to D_{l-1} or to D_{l+1}. This follows from the fact: if two permutations differ by a transposition then the numbers of cycles in them differ by 1. Then clearly

$$|f(u) - f(v)| \leq 2 \max_l \{|D_l|\}.$$

Finally, we estimate $|D_l|$. Each permutation $\pi \in T(G)$ can be expressed as a product $\pi_1 \pi_2 ... \pi_k$, where $\pi_i \in T(G_i), i = 1, 2, ..., k$. If a permutation $\pi \in T(G)$ has exactly l cycles then the permutation π_i has x_i cycles, for $i = 1, 2, ..., k$, where $1 \leq x_i \leq n_i$ and $x_1 + x_2 + ... + x_k = l$. It is known that there are $s(n_i, x_i)$ permutations of n_i elements having exactly x_i cycles [14]. Hence for a given k-tuple $x_1, x_2, ...x_k$, there are $\prod_{i=1}^{k} s(n_i, x_i)$ possible permutations of n elements having l cycles. Summing this over all permissible partition of l into $x_1, x_2, ...x_k$ we get the claimed estimation for $|D_l|$. $\square$

Corollary 5.1 *For bandwidths of the star graph and the complete transposition graph we have*

$$band(S_n), band(CT_n) \leq 2\max_l\{s(n,l)\} = O\left(\frac{n!}{\sqrt{\log n}}\right).$$

The rightmost estimation follows from [4]. The well known lower bound argument [2] shows that $band(G) \geq (|V_G|-1)/diam(G)$, where $diam(G)$ denotes the diameter of G. Diameters of the star and complete transposition graphs are of order $\Theta(n)$ which implies lower bounds for bandwidths of order $\Omega((n-1)!)$.
The n-dimensional hypercube is an example that the bound in Theorem 5.1 can not be improved in general.

Corollary 5.2 *The bandwidth of the m-dimensional hypercube satisfies*

$$band(Q_m) \leq 2\binom{m}{\lfloor \frac{m}{2} \rfloor} = O\left(\frac{2^m}{\sqrt{m}}\right).$$

Proof. As the m-dimensional hypercube is a transposition graph of a graph consisting of m independent edges, we have $n_i = 2, n = 2m, k = m$ in Theorem 5.1. Hence

$$\begin{aligned} band(Q_m) &\leq 2\max_l \sum_{\substack{1\leq x_i \leq 2, 1\leq i\leq m\\ x_1+x_2+\ldots+x_m=l}} \prod_{i=1}^{m} s(2,x_i) = 2\max_l \sum_{\substack{1\leq x_i \leq 2, 1\leq i\leq m\\ x_1+x_2+\ldots+x_m=l}} 1 \\ &= 2\max_l \binom{m}{l-m} = 2\binom{m}{\lfloor \frac{m}{2} \rfloor} = O\left(\frac{2^m}{\sqrt{m}}\right). \quad \square \end{aligned}$$

This is asymptotically optimal as Harper [5] proved that $band(Q_m) = \sum_{i=0}^{m}\binom{i}{\lfloor i/2 \rfloor} = \Theta(\frac{2^m}{\sqrt{m}})$.

6 Conclusions

We studied bisection widths of transposition graphs. In particular, we proved asymptotically optimal bisection width of the complete transposition graph and nearly exact value of the bisection width of the star graph. The first result solves an open problem of Leighton [10]. The results have consequences for optimal VLSI layouts, cutwidths and crossing numbers of transposition graphs. We also give a general upper bounds for bandwidths of transposition graphs. Our paper leaves a few open problems.
Applying our approach to finding bisection width of the bubblesort and modified bubblesort graph we get upper and lower bounds differing by multiplicative factors of n^2 and n, respectively.
Another important parameter of interconnection networks (graphs) is the vertex bisection width, i.e. the smallest vertex cut that divides a graph into roughly equal parts. It is known that the vertex bisection width is no greater than the bandwidth. On the other hand, the bisection width is at most the vertex bisection width $\times$ the maximum degree of the graph. These observations give the same lower and upper bounds for the vertex bisection widths of the star and complete transposition graphs as those for bandwidths (Corollary 5.1).
An important open problem is to find the best possible bound for $d_G(\pi, id)$ in transposition graphs. In particular, the diameter of the modified bubblesort graph is not known, although there exists a polynomial time algorithm for determining the shortest path [6].

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