Broadcasting from Multiple Originators

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Abstract

We begin an investigation of broadcasting from multiple originators, a variant of broadcasting in which any k vertices may be the originators of a message in a network of n vertices. The requirement is that the message be distributed to all n vertices in minimum time. A *minimum k-originator broadcast graph* is a graph on n vertices with the fewest edges such that any subset of k vertices can broadcast in minimum time. $B_k(n)$ is the number of edges in such a graph. In this paper, we present asymptotic upper and lower bounds on $B_k(n)$. We also present an exact result for the case when $k \geq \frac{n}{2}$. We also give an upper bound on the number of edges in a relaxed version of this problem in which one additional time unit is allowed for the broadcast.

keywords: broadcast, multiple originator, minimum broadcast graph, relaxed broadcast

1 Introduction

Broadcasting is the process of message dissemination in a communication network in which a message, originated by one vertex, is transmitted to all vertices of the network by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible. Typically, it is assumed that each call involves only one informed vertex and one of its neighbors, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call its neighbors. Here, we consider broadcasting from any set of k originators.

Given a connected graph $G = (V, E)$ and a subset of the vertices $V' \subseteq V$ the *k-originator broadcast time of the set* V', $b(V')$, is the minimum number of time units required to complete a broadcast from the vertices V'. Since the number of informed vertices can at most be doubled during each time unit, it is clear that for any set V' of k vertices in a connected graph G with n vertices, $b(V') \geq t(n, k)$, where $t(n, k) = \lceil \log_2 \frac{n}{k} \rceil$. The *$k$-originator broadcast time of a graph* G, denoted $b_k(G)$, is the maximum broadcast time of any such subset V' in G, with $|V'| = k$, i.e. $b_k(G) = \max\{b(V')|V' \subseteq V, |V'| = k\}$. We use the term *$k$-originator broadcast graph* to refer to any graph G on n vertices with $b_k(G) = t(n, k)$. The *k-originator broadcast function*, $B_k(n)$, is the minimum number of edges in any k-originator broadcast graph on n vertices. A *minimum k-originator broadcast graph* is a k-originator broadcast graph on n vertices having $B_k(n)$ edges.

Garey and Johnson [10] list the following problem in their list of NP-complete problems: Given $G = (V, E)$, subset $V' \subseteq V$, and positive integer t, can a message be broadcast from V' to V in time t? Very little has been done on this problem for multiple originators, that is, when $|V'| > 1$. Hedetniemi and Hedetniemi [16] considered a related problem in trees. In particular, they investigated the number of originators required to broadcast in a tree in a fixed number of time units. They solved the problem for trees with time = 1 or 2. More recently, Chia, Kuo, and Tung [4] considered multiple originator broadcasting in paths, grids, and in products of graphs. They studied the function $\min\{b(V')|V' \subseteq V, |V'| = k\}$ (rather than max) which corresponds to the broadcast time of vertices commonly referred to as the "broadcast center" of the graph [25].

Otherwise, the work on this problem has focussed on the special case of a single originator, that is, when $|V'| = 1$. For a single originator, Johnson showed that it is NP-complete to determine whether an arbitrary vertex in a graph G on n vertices can broadcast in time $\lceil \log_2 n \rceil$ (see page 190 of [7]). The study of minimum 1-originator broadcast graphs and the function $B_1(n)$ has a long history. Exact values of $B_1(n)$ are known only for $n = 2^k$ (where $k \geq 1$), $n = 2^k - 2$ (where $k \geq 2$), and for several specific values of n. Upper bounds on $B_1(n)$ are obtained by constructing 1-originator broadcast graphs. A long sequence of papers have presented techniques to construct 1-originator broadcast graphs for large n (see, for example, [1,3,5,7,11–14,20,23,26,27]). In addition to these constructions, a few papers have presented lower bounds on $B_1(n)$ [11,12,23,24]. See [5] for more details on the historical development of this search. It has long been conjectured that $B_1(n)$ is monotonic between consecutive powers of 2. A partial result in this direction has been published recently [15].

For surveys of results on broadcasting and other related problems, see Hedetniemi, Hedetniemi and Liestman [17], Fraigniaud and Lazard [9], Hromkovič, Klasing, Monien, and Peine [18], and Hromkovič, Klasing, Pelc, Ružička, and Unger [19].

In this paper, we initiate the study of minimum k-originator broadcast graphs. We are motivated, in part, by the historical interest in this more general problem. Although it is unclear that this will shed light on the heavily studied 1-originator problem, it is not unreasonable to assume that a message to be broadcast may be started at multiple locations. For example, suppose that you have a backbone network with a set of vertices that are more reliably connected. We could expect those vertices to agree on the information that must be broadcast to the other vertices.

2 Asymptotic Bounds

We begin this section with a derivation of general lower bounds for $B_k(n)$. First we present a general result on how the degree of an originator affects broadcasting.

Lemma 2.1 *For any originator v, with degree d_v, the number of nodes informed after c time units is at most $2^c - 2^{c-d_v} + 1$.*

PROOF If $d_v > c$ the result follows by trivial bounds on repeated doubling. For $d_v \leq c$ we will prove the result by induction on $c - d_v$. The basis, $d_v = c$, also follows by repeated doubling. Suppose $d_v < c$, so v is idle some time during the first c time units; idle in the sense that it is not involved in a call or is making a repetitious call. Let t_v be the first such idle time, so $t_v \leq d_v + 1 \leq c$.

Let $f_v(i)$ for $i = 0, 1, \ldots, c$ be the number of vertices that know the message at time i of the broadcast scheme such that the message they received (for the first time) originated in v. Clearly, $f_v(0) = 1$ and $f_v(i) \leq 2f_v(i-1)$ for any $i \geq 1$. Hence $f_v(t_v - 1) \leq 2^{t_v - 1}$, however $f_v(t_v) \leq 2f_v(t_v - 1) - 1 \leq 2^{t_v} - 1$. In each of the remaining $c - t_v$ steps, each of $f_v(t_v) - 1$ informed vertices other than v can at most double. Further, by the inductive hypothesis, during the same $c - t_v$ time units v continues broadcasting with an effective degree of $d_v - t_v + 1$. Hence

$$f_v(c) \leq (2^{t_v} - 1 - 1)2^{m-t_v} + (2^{m-t_v} - 2^{(c-t_v)-(d_v-t_v+1)} + 1) \leq 2^c - 2^{c-d_v} + 1.$$

□

Let $L(x)$ be the number of leading 1 bits in the binary representation of the integer x. That is, if $x = 2^m$ then $L(x) = 1$ and if $2^{m-1} < x < 2^m$ then $L(x) = l$ when $2^m - 2^{m-l} < x \leq 2^m - 2^{m-l-1}$.

Theorem 2.2 *For every* $n \geq 1$, B_k *is* $\Omega\left(L(\lceil \frac{n}{k} \rceil - 1) \cdot (n-k)\right)$.

PROOF First, suppose $2^{m-1} < \lceil \frac{n}{k} \rceil < 2^m$. Assume that $l = L(\lceil \frac{n}{k} \rceil) > 1$, since otherwise the theorem just states that a linear number of edges are needed, which is clear. Note that only $t(n,k) = m$ time units are allowed. Suppose there exists a set $Z \subseteq V$ of k vertices, all with degree less than $l-1$. By the previous lemma, each originator $v \in Z$ can inform at most

$$2^m - 2^{m-d_v} + 1 < 2^m - 2^{m-l+1} + 1 < \lceil \tfrac{n}{k} \rceil - 1,$$

using the fact that $d_v + 1 < l$. Therefore, since the k originators cannot inform n vertices in m time units, at most $k-1$ vertices can have degree less that l. To be consistent with next case without affecting the asymptotic result, we choose to replace $L(\lceil \frac{n}{k} \rceil)$ with $L(\lceil \frac{n}{k} \rceil - 1)$ since $1 \leq L(\lceil \frac{n}{k} \rceil) - 1 \leq L(\lceil \frac{n}{k} \rceil - 1) \leq L(\lceil \frac{n}{k} \rceil)$.

Second, suppose $\lceil \frac{n}{k} \rceil = 2^m$, so that $t(n,k) = m$. Suppose there exists a set $Z \subseteq V$ of k vertices, all with degree less than m. As above, it can be shown that any $v \in Z$ can inform at most

$$2^m - 2^{m-d_v} + 1 \leq 2^m - 2^{m-(m-1)} + 1 \leq 2^m - 1 = \lceil \tfrac{n}{k} \rceil - 1.$$

So, again, the originators in Z cannot inform n vertices and $n-k+1$ vertices must have degree at least m. Since in this second case $L(\lceil \frac{n}{k} \rceil) = 1$, and $L(\lceil \frac{n}{k} \rceil - 1) = m$, in the theorem we use $L(\lceil \frac{n}{k} \rceil - 1)$. □

The following corollary was also shown by Grigni and Peleg [12].

Corollary 1 *For every* $n \geq 1$, $B_1(n)$ *is* $\Omega\left(L(n-1) \cdot n\right)$.

We now present a derivation of a general upper bound for $B_k(n)$. We give a construction that has two variants.

Lemma 2.3 *If* $2^{m-1} < \lceil \frac{n}{k} \rceil < 2^m$ *then* $B_k(n)$ *is* $O\left(L(\lceil \frac{n}{k} \rceil) \cdot n \cdot k\right)$.

PROOF As in the last section, let $l = L(\lceil \frac{n}{k} \rceil)$, so that $2^m - 2^{m-l} < \lceil \frac{n}{k} \rceil \leq 2^m - 2^{m-l-1}$. We construct a k-originator broadcast graph $H(n,k)$ on n vertices. Our method builds on the construction of a similar graph by Grigni and Peleg [12]. Let $h_i = 2^{m-i}$ for $i = 1, 2, \ldots, l$ and let $h_{l+1} = \lceil \frac{n}{k} \rceil - (h_1 + h_2 + \cdots + h_l)$. A simple calculation shows that $h_{l+1} \leq 2^{m-(l+1)}$. The broadcast graph $H(n,k)$ contains k copies of trees T_{h_i} (as described above) for $1 \leq i \leq l+1$. If $k \cdot \sum_{i=1}^{l+1} h_i > n$, we replace some copies of $T_{h_{l+1}}$ with copies of $T_{h_{l+1}-1}$ so that the order of $H(n,k)$ is n; however, without loss of generality, we still refer to these trees as $T_{h_{l+1}}$. Finally, we join every vertex of these trees to each of the $k(l+1)$ roots by an edge, except that we do not add loops and parallel edges.

It is easy to see that the graph $H(n,k)$ has $O(lnk)$ edges. It remains to show that k-originator broadcasting in $H(n,k)$ can be done in time $t(n,k) = m$. Let $A \subseteq V$ be the set of k originators in $H(n,k)$. We group the trees T_{h_i} into k *stands* so that each stand contains exactly one tree of each size, and at most one tree in each stand contains some vertices of A. This is always possible, since $|A| = k$ and there are k copies of each tree T_{h_i}. Each stand contains $l+1$ trees. In each stand, we order the trees from largest to smallest. We order the stands so that stand j follows stand i if j's originators are in a tree smaller than the tree containing originators of stand i, or if stand j does not contain any originators. Note that in general this is not a unique ordering. Let $u_1, u_2, \ldots, u_k$ be an enumeration of the originators in A so that u_i precedes u_j if u_i is in a stand preceding the stand of u_j, or u_i and u_j are in the same stand (and, hence, in the same tree of that stand) and u_i is not informed after u_j using the tree policy in that tree. Again, this is in general not unique. We let p denote the number of stands containing originators, that is, $k - p$ stands contain no originator.

We distinguish trees in stand j, $1 \leq j \leq k$, by adding a superscript j, i.e.

$$T^j_{h_1}, T^j_{h_2}, \ldots, T^j_{h_{l+1}}$$

are trees in stand j, and their respective roots are $r^j_1, r^j_2, \ldots, r^j_{l+1}$. Each originator u_i appears in some tree $T^s_{h_j}$, that is, it is in the j-th largest tree of the stand s. We say that $\textsc{tree}(u_i) = j$ and $\textsc{stand}(u_i) = s$. We also use $\textsc{time}(u_i)$ to denote the time (number of time steps) needed for u_i to receive the message broadcast from r^s_j using the tree policy scheme in $T^s_{h_j}$. Each originator u_i is responsible for informing the vertices in stand S_i. This will guarantee that if each originator will inform all vertices in the stand assigned to it, then all vertices of $H(n,k)$ will be informed. If u_i is the originator with smallest i among all originators in $\textsc{stand}(u_i)$ then $S_i = \textsc{stand}(u_i)$, otherwise, $S_i = p + i - \textsc{stand}(u_i)$.

We will make use of the following two broadcast tasks described by Grigni and Peleg [12]. The first, denoted by $A_1(j,s)$, is that of broadcasting a message initially stored in the root r^s_j to the vertices of the tree $T^s_{h_j}$, using only edges of that tree. (This is done using the tree policy.) The second task, denoted by $A_2(j,s)$, is that of broadcasting a message initially stored in the root r_j^s to all vertices of trees $T_{h_j}^s, \ldots, T_{h_{l+1}}^s$, using only edges incident to those vertices (i.e., without outside help). (This task has each root call the next smaller tree before broadcasting in its own tree.) It is easy to show (see [12] for details) that

Claim 2 *Task $A_1(j,s)$ is achievable in $m-j$ time units for every $1 \leq j \leq l+1$ and $1 \leq s \leq k$.*

Claim 3 *Task $A_2(j,s)$ is achievable in $m-j+1$ time units for every $1 \leq j \leq l+1$ and $1 \leq s \leq k$.*

It remains to describe how to broadcast from an originator u_i to its stand S_i. Let $p(u_i)$ denote the parent of u_i in its tree if u_i is not the root of the tree, and let $p(u_i) = u_i$ otherwise. The broadcast scheme is an extension of the corresponding scheme of Grigni and Peleg [12]. We distinguish two cases. If $\text{TIME}(u_i) = 0$ then $u_i = r_{\text{TREE}(u_i)}^{\text{STAND}(u_i)}$ and u_i spends the first $\text{TREE}(u_i)-1$ time units sending the message to $r_1^{S_i}, r_2^{S_i}, \ldots, r_{\text{TREE}(u_i)-1}^{S_i}$ (in this order). It follows by the definition of S_i that $S_i = \text{STAND}(u_i)$. By our regrouping of trees, the tree $T_{h_{\text{TREE}(u_i)}}^{S_i}$ is the only tree in its stand containing originators, and hence $T_{h_1}^{S_i}, T_{h_2}^{S_i}, \ldots, T_{h_{\text{TREE}(u_i)-1}}^{S_i}$ do not contain originators. Thus, the root $r_j^{S_i}$, $1 \leq j \leq \text{TREE}(u_i)-1$, receives the message at time j and then it performs task $A_1(j, S_i)$ which is achievable in $m-j$ time units by Claim 2. After the first $\text{TREE}(u_i)-1$ steps, the originator $u_i = r_{\text{TREE}(u_i)}^{S_i}$ uses remaining $m - \text{TREE}(u_i) + 1$ time steps to perform task $A_2(\text{TREE}(u_i), S_i)$. This is achievable by Claim 3.

If $\text{TIME}(u_i) > 0$, then let $q = \min\{\text{TREE}(u_i) + \text{TIME}(u_i) - 1, l+1\}$. The originator u_i spends first q time units sending the message to $r_1^{S_i}, r_2^{S_i}, \ldots, r_q^{S_i}$ (in this order). If $S_i = \text{STAND}(u_i)$, then by our regrouping of trees, the tree $T_{\text{TREE}(u_i)}^{S_i}$ is the only tree in its stand containing originators, and hence $T_{h_1}^{S_i}, T_{h_2}^{S_i}, \ldots, T_{h_q}^{S_i}$ do not contain originators except possibly $T_{\text{TREE}(u_i)}^{S_i}$. However $r_{\text{TREE}(u_i)}^{S_i}$ is not an originator. Similarly, if $S_i \neq \text{STAND}(u_i)$, then by our regrouping of stands, the stand does not contain any originators, and hence $T_{h_1}^{S_i}, T_{h_2}^{S_i}, \ldots, T_{h_q}^{S_i}$ do not contain originators again. Thus, each root $r_j^{S_i}$, $1 \leq j \leq q$, receives the message in time j and has $m-j$ time units to complete the task $A_1(j, S_i)$. In particular, in the time step $\text{TREE}(u_i)+1$, the tree policy broadcast is started in the tree $T_{h_{\text{TREE}(u_i)}}^{S_i}$. If $l+1 \leq \text{TREE}(u_i) + \text{TIME}(u_i) - 1$, then the broadcast process is finished independently in each tree of the stand S_i by using the task $A_1(r_j^{S_i}, S_i)$ for $1 \leq j \leq q$. Hence we suppose $l+1 > \text{TREE}(u_i) + \text{TIME}(u_i) - 1$. At step $\text{TREE}(u_i) + \text{TIME}(u_i) = q+1$ the broadcast carried in $T_{h_{\text{TREE}(u_i)}}^{\text{STAND}(u_i)}$ is modified as follows: The parent $p(u_i)$ is supposed to send the message to u_i during this step. Instead, $p(u_i)$ sends the message to $r_{q+1}^{S_i}$ and u_i sends the message to $r_{q+2}^{S_i}$. (Note that $p(u_i)$ and u_i are not involved in any other part of the broadcast scheme at time $q+1$.) After this step, the vertices u_i and $p(u_i)$ resume their role as regular vertices in $T_{h_{\text{TREE}(u_i)}}^{\text{STAND}(u_i)}$. Furthermore, the remaining $m-(q+1)$ steps are used by $r_{q+1}^{S_i}$ and $r_{q+2}^{S_i}$ to perform tasks $A_1(q+1, S_i)$ and $A_2(q+2, S_i)$, respectively. By Claims 2 and 3 these tasks can be completed using $m-(q+1)$ steps and, thus, every vertex in the stand S_i will be informed. □

Lemma 2.4 *If* $\lceil \frac{n}{k} \rceil = 2^m$ *then* $B_k(n) \le O\left(m \cdot n \cdot k\right)$.

PROOF The construction here is similar to the previous proof, so the details are only sketched. The broadcast graph is formed from a forest of k trees, each tree is T_h, where each h is $\lceil \frac{n}{k} \rceil$ or $\lfloor \frac{n}{k} \rfloor$ so that there are a total n vertices. Further every vertex is connected to the root of every tree and to the children of every root. Multiple edges and loops are removed. Clearly the constructed graph has $O(mnk)$ edges, since each root of the forest has m children.

Again, let u_i be the ith originator. Let $\textsc{tree}(u_i)$ be the tree containing u_i and let $\textsc{time}(u_i)$ be the time unit that u_i would be informed using the tree policy in $\textsc{tree}(u_i)$. Let each u_i be assigned to a unique tree S_i; the assignment can be arbitrary except if that the root of a tree is assigned to that tree. The originator u_i is responsible for broadcasting to all of S_i.

For the first $\textsc{time}(u_i)$ time steps, u_i contacts the children of the root of S_i in increasing order of index. During time step $\textsc{time}(u_i)$ the parent of u_i (if there is one) informs the root of S_i. For the remaining steps u_i behaves like any other vertex of $\textsc{tree}(u_i)$ in broadcasting in that tree. It follows that after $m = t(n,k)$ time steps all vertices are informed. □

We can combine these two lemmas.

Theorem 2.5 *For every* $n > 1$, $B_k(n) \le O\left(L(\lceil \frac{n}{k} \rceil - 1) \cdot n \cdot k\right)$.

3 Other Results

When the broadcast must be completed in one time unit, we have an exact value for $B_k(n)$. As each vertex can participate in at most one call per time unit, a broadcast can be completed in one time unit only when $k \ge \frac{n}{2}$.

Theorem 3.1 $B_k(n) = \lceil \frac{n(n-k)}{2} \rceil$ *for* $k \ge \frac{n}{2}$.

PROOF Let $k \ge \frac{n}{2}$. If any vertex v is adjacent to fewer than $n-k$ vertices, by choosing k of the vertices not adjacent to v as originators, v can not be informed at time 1. Thus, in a k-originator broadcast graph, every vertex must be of degree at least $n-k$ and $B_k(n) \ge \lceil \frac{n(n-k)}{2} \rceil$.

For the upper bound, we consider three separate cases.

First, we consider even n. Construct an $n-k$ regular bipartite graph $G = (U \cup V, E)$ on n vertices where $U = \{u_0, u_1, \ldots, u_{\frac{n}{2}-1}\}$ and $V = \{v_0, v_1, \ldots, v_{\frac{n}{2}-1}\}$. Connect each u_i with $v_i, v_{i+1}, \ldots, v_{i+(n-k)-1}$ where the subscript calculations are performed modulo $\frac{n}{2}$.

It remains to show that any subset L of $n-k$ uninformed vertices can be matched to $n-k$ of the k originators. Consider the subgraph G' obtained as follows: Let $l \leq n-k$ be the number of vertices of U in L (leaving $(n-k)-l$ vertices of V in L) and let U' and V' denote $U \cap L$ and $V \cap L$, respectively. The vertices of G' are $U' \cup N_G(U') \cup V' \cup N_G(V')$. The edges of G' are those edges of the induced subgraph connecting members of L to non-members of L. That is, we only include the edges that may contribute to a matching between the uninformed vertices and the originators.

As the edges of G' are all between elements of $L = U' \cup V'$ and $R = (N_G(U') \setminus V') \cup (N_G(V') \setminus U')$ which are disjoint sets, G' is bipartite. By Hall's Theorem, we know that we can find a complete matching saturating every element of L (that is, we can broadcast to L in one time unit) iff for every subset S of L, $|N_{G'}(S)| \geq |S|$. Below, we simply use $N(W)$ to denote $N_{G'}(W)$.

Since the four sets U', V', $N(U')$, and $N(V')$ are pairwise disjoint, we need only show that for any subsets U'' of U' and V'' of V', $|N(U'')| \geq |U''|$ and $|N(V'')| \geq |V''|$.

In G, every vertex has $n-k$ neighbors and every subset of s vertices from U or from V has at least $\min\{(n-k)+(s-1), \frac{n}{2}\}$ neighbors.

In G', every vertex of U' with $|U'| = l$ has at least $(n-k)-(n-k-l) = l$ neighbors since there are $n-k-l$ members of L in V'. Further, for each subset U'' of s vertices of U' (where $1 \leq s \leq l$), $|N(U'')| \geq \min\{l+(s-1), \frac{n}{2}-(n-k-l)\} \geq \min\{l+s-1, l\} \geq s = |U''|$.

In G', every vertex of V' with $|V'| = n-k-l$ has at least $n-k-l$ neighbors since there are l members of L in U'. Further, for each subset V'' of p vertices of V' (where $1 \leq p \leq n-k-l$), $|N(V'')| \geq \min\{n-k-l+p-1, \frac{n}{2}-l\} \geq n-k-l = |V''|$.

Thus, $B_k(n) \leq \lceil \frac{n(n-k)}{2} \rceil$ when n is even.

For odd n, we have two cases depending on whether $n-k$ is even or odd.

When n is odd and $n-k$ is even, let $r = \frac{n-k}{2}$. The theorem statement can be rewritten as $B_k(n) = rn$ in this case. We construct a $2r$-regular graph G on vertices $V = \{0, 1, 2, ..., n-1\}$. Each vertex i is connected to vertices $i \pm 1, i \pm 2, ..., i \pm r$ where the calculations are performed modulo n.

As above, we show that any subset L of $2r = n - k$ uninformed vertices can be matched to $2r$ of the k originators.

Let L be any set of $2r$ uninformed vertices and consider the subgraph G' consisting of vertices $L \cup N_G(L)$ and only those edges between vertices in L and vertices in $N_G(L) \setminus L$.

Again, by Hall's Theorem, we know that we can find a complete matching saturating every element of L (that is, we can broadcast to L in one time unit) iff for every subset S of L, $|N(S)| \geq| S|$.

In G', each vertex of L has at least $2s - (|L| - 1) \geq 1$ distinct neighbors in $N(L) \setminus L$ and each subset of p vertices of L has at least $\min\{\lceil \frac{n}{2} \rceil, 2s - (|L| - p)\} \geq p$ distinct neighbors in $N(L) \setminus L$.

Thus, $B_k(n) \leq \lceil \frac{n(n-k)}{2} \rceil$ when n is odd and $n - k$ is even.

When n is odd and $n - k$ is odd, let $r = \lfloor \frac{n-k}{2} \rfloor$, so $2r + 1 = n - k$. We construct a graph G on vertices $V = \{0, 1, 2, ..., n - 1\}$ with $n - 1$ vertices of degree $2r + 1$ and one vertex of degree $2r + 2$. Each vertex i is connected to vertices $i \pm 1, i \pm 2, ..., i \pm r$. Further, each vertex i is connected to vertex i' where $i' = i + \lfloor \frac{n}{2} \rfloor$ when $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $i' = i + \lceil \frac{n}{2} \rceil$ when $\lceil \frac{n}{2} \rceil \leq i \leq n - 1$. Again, these calculations are performed modulo n.

As above, we show that any subset L of $2r + 1$ uninformed vertices can be matched to $2r + 1$ of the originators. We use Hall's Theorem again. In G', each subset of p vertices of L has at least $\min\{\lceil \frac{n}{2} \rceil, 2s + (p - 1) - (|L| - p)\} \geq p$ distinct neighbors in $N(L) \setminus L$.

Thus, $B_k(n) \leq \lceil \frac{n(n-k)}{2} \rceil$ when n is odd and $n - k$ is odd. □

It may sometimes be acceptable to allow an increase in k-originator broadcast time to decrease the number of edges in the graph. This led Farley [6], Liestman [21], Peleg [23], and Grigni and Peleg [12] to consider graphs in which broadcasting could be completed in slightly more than minimum time.

We say that G is a *k-originator relaxed broadcast graph* if the broadcast can be completed within $t(n, k) + 1$ time units for any set of k originators. Let $B'_k(n)$ denote the minimum number of edges in an k-originator relaxed broadcast graph on n vertices.

Let $k \geq 1$ and $n \geq k$ be two integers. A graph $G = (A \cup B; E)$ with $|A| = n$ and $|B| = k$ is called a (n, k)-*tunnel* if for any $A' \subseteq A$ with $|A'| = k$ the induced subgraph $G[A' \cup B]$ has a perfect matching. Note that when $A \cap B \neq \emptyset$ then the "perfect matching" might involve loop edges. The complete bipartite graph $K_{n,k}$ is a (n, k)-tunnel. Let $e_{n,k}$ be the minimum number of edges in any (n, k)-tunnel. The following simple lemma determines the value of

$e_{n,k}$.

Lemma 3.2 *For every $k \geq 1$ and $n \geq k$, $e_{n,k} = k(n-k+1)$.*

PROOF First, we show the lower bound. Suppose there is a vertex $x \in B$ incident on at most $n-k$ edges. Let $A'' \subseteq A$ be the set of vertices that are not adjacent to x. It follows that $|A''| \geq k$. Choose any $A' \subseteq A''$ such that $|A'| = k$. Now x is an isolated vertex in $G[A' \cup B]$, a contradiction.

Second, we construct a (n,k)-tunnel $G_{n,k} = (A \cup B; E)$ with $e_{n,k}$ edges. Let $n = k + l$, $A = \{a_1, a_2, \ldots, a_n\}$, and $B = \{b_1, b_2, \ldots, b_k\}$. We define

$$E = \{b_i a_i, b_i a_{i+1}, \ldots, b_i a_{i+l} \,:\, 1 \leq i \leq k\}.$$

The size of $G_{n,k}$ is $e_{n,k}$ as claimed. This edge set may contain parallel and loop edges. It is enough to show that $G_{n,k}$ is a (n,k)-tunnel. Let G' be the graph obtained from G by removing l vertices of A, say $a_{j_1}, a_{j_2}, \cdots, a_{j_l}$ where $j_1 < j_2 < \ldots < j_l$. Now

$$a_1b_1, a_2b_2, \ldots, a_{j_1-1}b_{j_1-1}, a_{j_1+1}b_{j_1}, a_{j_1+2}b_{j_1+1}, \ldots, a_{j_2-1}b_{j_2-2}, a_{j_2+1}b_{j_2-1}, \ldots, a_{k+l}b_k$$

is a perfect matching in G'. This shows that $G_{n,k}$ is a (n,k)-tunnel. □

We construct graph $G(n,k)$ with at most $(k+1)(n-k)+k$ edges and such that $b_k(G(n,k)) \leq t(n,k)+1$. Our construction is a modification of Peleg's construction [23]. Both constructions begin with a forest of rooted trees on n vertices. In Peleg's construction, each non-root vertex is joined by an edge to each of the roots. Our construction begins with a larger number of smaller trees divided into k subforests, but we still connect every vertex to the root of every tree. In our construction, we also use a class of trees constructed by Farley and Proskurowski [8] as well as the (n,k)-tunnel $G_{n,k}$.

For any $h \geq 1$, the tree T_h is constructed as follows: Vertices of T_h are indexed $1, 2, \ldots, h$ with 1 being the root; note that h is not the height of the tree. Further, every vertex i for $2 \leq i \leq h$ is joined to the vertex $i - 2^{\lceil \log_2 i \rceil - 1}$. (Note that when $h = 2^j$, T_h is known as a binomial tree, and if $h \neq 2^j$ it is a partial binomial tree.) As noted by Farley and Proskurowski [8], broadcasting in T_h from vertex 1 can be done in $t(h,1) = \lceil \log_2 h \rceil$ time. This can be achieved by a broadcast scheme where each informed vertex informs its children in order of increasing index. We will refer to this broadcast policy as the *tree policy* on T_h.

Theorem 3.3 *For every $k \leq n$, $B'_k(n) \leq (k+1)(n-k)+k$.*

PROOF We construct a graph $G(n,k)$ starting with k trees T_i for $i = \lfloor \frac{n}{k} \rfloor$ or $i = \lceil \frac{n}{k} \rceil$ so that these k trees have a total of n vertices. The forest has $n-k$ edges. Let $B = \{b_1, b_2, \dots, b_k\}$ be the set of roots of these trees, and let $A = \{a_1, a_2, \dots, a_n\}$ be the set of all vertices in these trees. Note that vertices of $B \subseteq A$. We connect these trees together using the edges of the (n,k)-tunnel $G_{n,k}$; note that each vertex in A is connected to vertices in B. Finally, we remove all parallel edges and loops that arise in the construction.

Let A' be any set of k vertices in $G(n,k)$. During the first step of the broadcast procedure, every vertex of A' sends its message to a selected vertex in B so that at the end of the first step every vertex of B knows the message. For this, we use a matching between A' and B in $G_{n,k}$. The scheme then proceeds with broadcasting in each copy of T_i. Note that for $1 \le k \le n$, $\left\lceil \log_2 \lceil \frac{n}{k} \rceil \right\rceil = \left\lceil \log_2 \frac{n}{k} \right\rceil$ so for each of the trees T_i the root can broadcast to the vertices of T_i in $t(n,k)$ time. Hence broadcasting in $G(n,k)$ is completed within $t(n,k)+1$ time units, as required. The number of edges in $G(n,k)$ is at most $k(n-k+1)+(n-k)$. $\square$

4 Conclusion

We have shown that for every $n \ge 1$, B_k is $\Omega\left(L(\lceil \frac{n}{k} \rceil - 1) \cdot (n-k)\right)$ and that $B_k(n) \le O\left(L(\lceil \frac{n}{k} \rceil - 1) \cdot n \cdot k\right)$. Although there is a gap between these bounds, we have no conjecture on either (or both) can be improved. Note for $k=1$, our bounds match the known bounds.

We have shown that when k-originator broadcasting can be completed in one time unit, $B_k(n) = \lceil \frac{n(n-k)}{2} \rceil$. In this case, we have exact values for all such n and k, but for when the k-originator broadcast time is greater than one, we have only partial results. These include loose bounds and exact values for small n and k and are included in a forthcoming paper [22].

Finally, we considered the problem of relaxed k-originator broadcasting and showed that for every $k \le n$, $B'_k(n) \le (k+1)(n-k)+k$. This result echoes the result from Grigni and Peleg [12] in that it shows that significantly fewer (removing the $L(\frac{n}{k})$ term) edges are necessary in the relaxed case.

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