On a generalization of Chvátal's condition giving new hamiltonian degree sequences

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Abstract

By imposing a structural criterion on a graph, we generalize the well-known Chvátal's sufficient condition for hamiltonicity [On Hamilton's ideals. *J. Combinatorial Theory Ser. B*, 12:163–168, 1972]. Using this result, we describe a new class of hamiltonian degree sequences which contains the sequences given by Chvátal's condition, as well as a class of degree sequences described by Fan and Liu [On the degree sequences ensuring the graphs to be Hamiltonian. *J. Systems Sci. Math. Sci.*, 4(1):27–32, 1984].

1 Introduction

A graph is denoted by $G = (V(G), E(G))$ where $V(G)$ is the set of vertices and $E(G)$ is the set of edges of G. We only consider simple graphs with $|V(G)| = n \geq 3$. Unless otherwise stated, we follow standard definitions and notation. By $G + H$ we denote the disjoint *union* of G and H, while by $G \vee H$ we denote the

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join of G and H obtained from $G+H$ by connecting every vertex of G with every vertex of H.

A *Hamilton cycle* in G is a cycle visiting every vertex of G exactly once. No non-trivial characterization of hamiltonian graphs, i.e. graphs containing a Hamilton cycle, is known. However, a number of sufficient conditions for the existence of such a cycle in a given graph are known (see [5, 6] for a survey). Several of these results are based on an edge-density argument. Two fundamental conditions in this direction are theorems of Chvátal [3] and Bondy and Chvátal [1].

Theorem 1 [3] *Let the degree sequence of* G *be* $d_1 \leq d_2 \leq \cdots \leq d_n$. *If*

$$d_k \leq k < \frac{n}{2} \;\Rightarrow\; d_{n-k} \geq n-k, \tag{1}$$

then G *is hamiltonian.*

Bondy and Chvátal defined a *closure*, $cl(G)$, of G as the graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree-sum is at least n, until no such pair remains. Using the concept of closure, and the fundamental fact that

If P is a Hamilton path in G with the degree-sum of its end-vertices equal to or greater than n, then G is hamiltonian. $(*)$

they proved a generalization of Theorem 1.

Theorem 2 [1] *A graph* G *is hamiltonian if and only if* $cl(G)$ *is hamiltonian.*

It can be shown that graphs satisfying Chvátal's condition have a complete closure, hence Theorem 1 follows from Theorem 2. However, the attractiveness of Theorem 1 is in the fact that it offers an elegant sufficient condition for *hamiltonian degree sequences.* A degree sequence is *hamiltonian* if every graph with this degree sequence is hamiltonian. Remarkably, only a few results are known that provide hamiltonian degree sequences not included in Theorem 1, and mostly they are on regular or almost regular graphs. Probably the most well-known is the result of Nash-Williams [8] on k-regular graphs.

Theorem 3 [8] *Every k-regular graph on $2k + 1$ vertices is hamiltonian.*

For 2-connected graphs an even weaker degree condition for hamiltonicity than that of Theorem 3 was proved by Jackson in [7]. Note that by imposing stronger connectivity conditions, the degree-bound can be further lowered, cf. [2].

Theorem 4 [7] *Every 2-connected k-regular graph on at most $3k$ vertices is hamiltonian.*

As every hamiltonian graph is 2-connected, Theorem 3 follows from Theorem 4. For almost regular graphs, we mention two interesting results by Fan and Liu.

Theorem 5 [4] *Let G be a graph with degree sequence satisfying*

$$d_i = \begin{cases} r - t, & 1 \leq i \leq r, \\ n - r + t - 1, & r < i \leq n \end{cases}$$

where r and t are integers such that $2 \leq 2t < r < (n + 2t)/3$. Then G is hamiltonian.

Theorem 6 [4] *Let G be a graph with degree sequence satisfying*

$$d_i = \begin{cases} r - t, & 1 \leq i \leq l, \\ r, & l < i \leq r, \\ n - r - 1, & r < i \leq n \end{cases}$$

where r, l, and t are integers such that $1 \leq l \leq r - 1 < \lfloor n/2 \rfloor - 1$ and $0 \leq t < \min\{r/l, r - l\}$. Then G is hamiltonian.

2 Results

In this paper, we first show that Chvátal's degree condition (1) can be weakened provided that we assume some structural properties of the graph G. In particular, we will assume 2-connectivity of G and non-membership of G in some class of non-hamiltonian graphs. Since hamiltonian graphs have both these properties, our degree condition will generalize Theorem 1. Then we will study what requirements on the degree sequence of G will guarantee the two structural properties of G. By putting together all these requirements on degree sequences, we will determine some new hamiltonian degree sequences. Our first result is the following theorem.

Theorem 7 *Let G be 2-connected with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose there exists an integer k, $2 \leq k < n/2$, such that*

(i) $d_i \geq i+1$ *for* $1 \leq i \leq k-1$,

(ii) $d_k \geq k$, *and*

(iii) $d_{k+1} \geq k+1 \quad$ *and* $\quad d_i \leq i \Rightarrow d_{n-i} \geq n-i$ *for* $\; k+1 \leq i < \frac{n}{2}$.

Then G is either hamiltonian or a subgraph of $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$ where $\min\{i : d_i = i+1\} \leq j \leq k-1$.

A graph G is *t-tough* if for any separating set $S \subseteq V(G)$ the number of components of $G - S$ is at most $|S|/t$. Note that every hamiltonian graph is 1-tough, but not every 1-tough graph is hamiltonian. Since no spanning subgraph of the graph $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$ from Theorem 7 is 1-tough, we have the following corollary.

Corollary 8 *Let G be a 1-tough graph with a degree sequence satisfying* (i)-(iii) *in the statement of Theorem 7. Then G is hamiltonian.*

As we already mentioned, Theorem 7 generalizes Theorem 1. Moreover, there are graphs which satisfy all conditions of Corollary 8, and whose closure equals the original graph. Hence Theorem 2 cannot be used to certify their hamiltonicity. As an example, consider the graph $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$ for some $2 \leq j \leq k-3 < n/2-3$, in which you delete an edge in the copy of K_{k-j} and join the two end-vertices of the deleted edge to a single vertex in the copy of $\overline{K_j}$.

Proof of Theorem 7. Let us consider a 2-connected non-hamiltonian graph G with a degree sequence that satisfies (i) - (iii) for some fixed k, $2 \le k < n/2$. Suppose, by way of contradiction, that G is not a subgraph of $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$ for any $m_G \le j \le k-1$, where $m_G = \min\{i : d_i = i+1\}$. Note that it is easy to see that m_G exists.

Adding an edge into G results in a new graph G^* which is 2-connected and its degree sequence satisfies (i) - (iii) for the original k. Since the minimum degree of G is at least two, we have $m_G \le m_{G^*}$. Thus, G^* cannot be a subgraph of $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$ for any $m_{G^*} \le j \le k-1$. Indeed, $G \subseteq G^*$ and G is not a subgraph of such graphs by our assumption. (Recall that (i) - (iii) are satisfied for the same k in both G and G^*, and $m_G \le m_{G^*}$.)

Hence we may suppose G has maximum possible number of edges such that it has no Hamilton cycle. It follows that every pair of non-adjacent vertices in G is joined by a Hamilton path.

If $d_k \ge k+1$ then (i) - (iii) imply hamiltonicity of G by Theorem 1, thus we have $m_G \le k-1$. Let $V(G) = \{v_1, \ldots, v_n\}$ so that $d(v_i) = d_i$ for $1 \le i \le n$ (where $d(v_i)$ denotes the degree of the vertex v_i). We partition the vertex set of G into three sets as follows

$$
\begin{aligned}
V_1 &= \{v_1, v_2, \ldots, v_k\}, \\
V_2 &= \{x \in V(G) - V_1 : \exists y \in V_1, xy \in E(G)\}, \\
V_3 &= V(G) - (V_1 \cup V_2).
\end{aligned}
$$

We claim that the graph induced by vertices in $V_2 \cup V_3$ is a clique. Assume not, and let $x, y \in V_2 \cup V_3$ be a pair of non-adjacent vertices such that $d(x) + d(y)$ is maximum possible with $d(x) \le d(y)$. By the maximality of G, there is an xy Hamilton path $(x = w_1, w_2, \ldots, w_n = y)$, and now by the above fact $(*)$, we have $d(x) + d(y) \le n-1$. Let $d(x) = m$. If $w_1 w_{i+1} \in E(G)$ for some $2 \le i \le n-1$, then $(w_i, w_{i-1}, \ldots, w_1, w_{i+1}, w_{i+2}, \ldots, w_n = y)$ is a $w_i y$ Hamilton path, and hence $w_i y \notin E(G)$ and $d(w_i) \le m$. Consequently, there are m vertices in G with degree at most m, which implies $d_m \le m < n/2$. Condition (iii) implies $m \ge k+1$, and $d_{n-m} \geq n-m$, again by (iii). Hence there exist $m+1$ vertices with degree at least $n-m$. At least one of these, say y', is not adjacent to x. However, $d(x)+d(y') \geq m+(n-m)=n$ which contradicts the choice of x and y. This proves that $V_2 \cup V_3$ is indeed a clique, and hence $d_{k+1} \geq n-k-1$.

If $|V_2|=1$ then G has a cut-vertex, a contradiction. So we can assume $|V_2| \geq 2$. We claim that every vertex in V_1 is adjacent to every vertex in V_2. Suppose not, and let $1 \leq i \leq k$ be as large as possible such that $v_i \in V_1$ is not adjacent to some vertex $x \in V_2$. If $i=k$, then $d(x) \geq n-k-1+1=n-k$ by the definition of V_2. Since $d(v_k) \geq k$ by (ii), a contradiction follows from the fundamental fact $(*)$ above. If $i<k$, then $d(x) \geq n-k-1+k-i=n-i-1$ and $d(v_i) \geq i+1$ by (i), again a contradiction.

If $|V_2|>k$ then G is hamiltonian, a contradiction. Therefore $|V_2|=l$ for some $2 \leq l \leq k$. Let G_1 be the graph induced by all vertices in V_1. It is easy to verify that G is hamiltonian, if and only if the vertices of G_1 can be covered by at most $l-1$ pairwise vertex disjoint paths. Furthermore, the latter holds, if and only if $G'=G_1 \vee K_{l-1}$ is hamiltonian.

Let $d_1' \leq d_2' \leq \cdots \leq d_{n'}'$ be the degree sequence of G'. It follows that $n'=k+l-1$, $d_i'=d_i-1$ for $1 \leq i \leq k$, and $d_i'=k+l-2$ for $k+1 \leq i \leq n'$. Since G is not hamiltonian, G' is not hamiltonian either. Therefore, by Theorem 1, there exists $1 \leq t<(k+l-1)/2$ such that $d_t' \leq t$ and $d_{k+l-1-t}' \leq k+l-t-2$. From the latter, we have $k+l-1-t \leq k$. From $d_i'=d_i-1$ $(1 \leq i \leq k)$, we have $d_t \leq t+1$ and $d_{k+l-1-t} \leq k+l-1-t$. The latter (together with $k+l-1-t \leq k$) implies $k+l-1-t=k$, that is $t=l-1$. Therefore, $d_{l-1} \leq l$, which implies $d_{l-1}=l$ by (i).

Since every vertex in V_1 is adjacent to every vertex in V_2, $d_1 \geq l$, and hence $d_1=d_2=\cdots=d_{l-1}=l$, and the vertices $v_1, \ldots, v_{l-1}$ are isolated in G_1. Let $j=l-1$. We find that $m_G=j \leq k-1$ and the deletion of the $j+1$ vertices of V_2 splits G into $j+2$ components, j of which consist of a single vertex only and the orders of the other two are $k-j$ and $n-k-j-1$, respectively. This is another way of saying that G is a subgraph of $K_{j+1} \vee \left(\overline{K_j}+K_{k-j}+K_{n-k-j-1}\right)$, a contradiction. The proof is complete. $\square$

In our next result, we provide a degree condition that guarantees hamiltonian degree sequences. This is achieved by transforming the 2-connectivity requirement from Theorem 7 into condition (v) as well as the non-membership in the class of spanning subgraphs of $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$ into condition (iv) in the corollary.

Corollary 9 *Suppose there exists an integer k, $2 \leq k < n/2$, such that a degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ satisfies*

(i) $d_i \geq i+1$ *for* $1 \leq i \leq k-1$,

(ii) $d_k \geq k$,

(iii) $d_{k+1} \geq k+1$ *and* $d_i \leq i \Rightarrow d_{n-i} \geq n-i$ *for* $k+1 \leq i < \frac{n}{2}$,

(iv) *for all j such that* $\min\{i : d_i = i+1\} \leq j \leq k-1$

$$\sum_{i=1}^{n-j-1} d_i > \sum_{i=n-j}^{n} d_i + 2\binom{k-j}{2} + 2\binom{n-k-j-1}{2}, \text{ and}$$

(v) $d_{n-1} \geq n-k$ *or* $d_n \leq n+k-w$
where $w = \min\{i : d_i = k\} + \min(\{n\} \cup \{i : d_i \geq n-k-1\})$.

Then $d_1 \leq d_2 \leq \cdots \leq d_n$ is hamiltonian.

Proof. Suppose $d_1 \leq d_2 \leq \cdots \leq d_n$ is a degree sequence which satisfies (i)-(v), and let G be a graph on this sequence. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ so that $d(v_i) = d_i$ $(1 \leq i \leq n)$. According to Theorem 7, it is enough to show that G is 2-connected and is not a subgraph of $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$ for any $m_G \leq j \leq k-1$. First suppose that $x \in V(G)$ is a cut-vertex of G. Let the vertex sets of the two (not necessarily connected) components of $G - x$ be V_1 and V_2 with $p = |V_1| \leq |V_2| = q$, respectively. Since the vertices of V_1 have degrees not exceeding p, we have $d_p \leq p < n/2$. Hence $p \geq k$ by (i). Moreover, $p \geq k+1$ is impossible since in this case $d_{n-p} \geq n-p = q+1$ would hold by (iii), contradicting the fact that all vertices but x have degrees not exceeding q. So $p = k$, $q = n-k-1$ and $V_1 = \{v_1, v_2, \ldots, v_k\}$. This implies $d_{n-1} \le n-k-1$, so the first part of condition (v) cannot be true. Denote $w_1 = \min\{i : d_i = k\}$ and $w_2 = \min(\{n\} \cup \{i : d_i \ge n-k-1\})$. Then all $k - w_1 + 1$ vertices of V_1 of degree k must be adjacent to x. If $w_2 < n$ then there are $n - w_2 + 1$ vertices of degree at least $n-k-1$, and hence at least $n - w_2$ of them must be in V_2. The latter is trivially true even if $w_2 = n$, so we conclude that $n - w_2$ vertices of V_2 must be adjacent to x. Therefore $d_n \ge d(x) \ge k - w_1 + 1 + n - w_2 = n + k - w + 1$. This contradicts (v), so G is 2-connected.

Now suppose that G is a subgraph of $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$. Let V' and V'' be the sets of vertices in G corresponding to vertices of K_{j+1} and $\overline{K_j} + K_{k-j} + K_{n-k-j-1}$, respectively. There are at least $\sum_{v \in V''} d(v) - 2\binom{k-j}{2} - 2\binom{n-k-j-1}{2}$ edges of G with one endvertex in V'' and the other in V'. Hence we have

$$\begin{aligned}
\sum_{i=n-j}^{n} d_i &\ge \sum_{v \in V'} d(v) \\
&\ge \sum_{v \in V''} d(v) - 2\binom{k-j}{2} - 2\binom{n-k-j-1}{2} \\
&\ge \sum_{i=1}^{n-j-1} d_i - 2\binom{k-j}{2} - 2\binom{n-k-j-1}{2},
\end{aligned}$$

which contradicts (iv) if $m_G \le j \le k-1$. Thus, for all these values of j, G is not a subgraph of $K_{j+1} \vee \left(\overline{K_j} + K_{k-j} + K_{n-k-j-1}\right)$, and hence must be hamiltonian by Theorem 7. $\square$

The non-natural look of the conditions (iv) and (v) may indicate that any deviation from Chvátal's hamiltonian degree sequences is not an easy problem. However, all conditions in Corollary 9 can be verified in polynomial time and the corollary is powerful enough to contain Theorem 6.

Proposition 10 *Theorem 6 follows from Corollary 9.*

Proof. Let $\mathcal{D} := d_1 \leq d_2 \leq \cdots \leq d_n$ be a degree sequence from Theorem 6. Since $t < r - l$, we have $l < r - t$ and hence $k = r$ and so (i) and (ii) hold for $\mathcal{D}$. Since $r < \lfloor \frac{n}{2} \rfloor$, we have $n - r - 1 > \lceil \frac{n}{2} \rceil - 1$, and so (iii) holds for $\mathcal{D}$. We have $\min\{i : d_i = k\} \leq l + 1$ and $\min(\{n\} \cup \{i : d_i \geq n - k - 1\}) = k + 1$, hence $n+k-w \geq n+k-(l+1+k+1) = n-l-2 \geq n-r-1 = d_n$ and so (v) holds for $\mathcal{D}$. It remains to show that (iv) holds for $\mathcal{D}$. We calculate the interval of feasible values for j (for which we have to check (iv)), i.e. we find $J = \min\{i : d_i = i+1\}$. By $t < r - l$ either $J = r - 1$ (if $t < r - l - 1$) or $J = l$ (if $t = r - l - 1$). Thus it always suffices to check (iv) for $l \leq j \leq r - 1$. Now (iv) is implied by

$$(r-t) \cdot l + r \cdot (r-l) + (n-r-1) \cdot (n-j-r-1) \; > \\ (n-r-1) \cdot (j+1) + (r-j) \cdot (r-j-1) + (n-r-j-1) \cdot (n-r-j-2).$$

Since $t < \frac{r}{l}$, we have $(r-t) \cdot l > (l-1) \cdot r$ and hence the above strict inequality is implied by

$$r(r-1) + (n-r-1) \cdot (n-j-r-1) \; \geq \\ (n-r-1) \cdot (j+1) + (r-j) \cdot (r-j-1) + (n-r-j-1) \cdot (n-r-j-2),$$

which is always satisfied if $0 \leq j \leq r-1$. This proves that Corollary 9 generalizes Theorem 6. $\square$

A natural question is whether Corollary 9 also generalizes Theorem 5. The answer is no. One of the reasons is the condition $d_{k+1} \geq k + 1$ in Corollary 9. We cannot show that this condition is necessary there, but the following example shows that the condition is necessary in Theorem 7. Let G_k be the graph with $V(G_k) = \{v_1, v_2, \ldots, v_{k+3}\} \cup \{u_1, u_2, \ldots, u_{k-1}\}$ and $E(G_k) = \{u_i u_j : i \neq j\} \cup \{v_i u_j : 1 \leq i \leq k+3, 1 \leq j \leq \min(i+2, k-1)\} \cup \{v_{k-2}v_{k-1}, v_k v_{k+1}, v_{k+2}v_{k+3}\}$. The graph is obviously non-hamiltonian, since after removing the $k-1$ vertices u_i, there are k components left. By an inspection of the degree sequence of G_k, we see that

$\min\{i : d_i = i + 1\} = k - 1$, hence $j = k - 1$ in the statement of the theorem. However, G_k is not a subgraph of $K_k \vee \left(\overline{K}_{k-1} + K_1 + K_2\right)$ since G_k does not have any independent set of size $k + 1$. On the other hand, G_k satisfies all conditions in (i)-(iii) of Theorem 7 except that $d_{k+1} = k$.

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