Closure for the Property of Having a Hamiltonian Prism

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ABSTRACT

We prove that a graph G of order n has a hamiltonian prism if and only if the graph $\mathrm{Cl}_{4n/3-4/3}(G)$ has a hamiltonian prism where $\mathrm{Cl}_{4n/3-4/3}(G)$ is the graph obtained from G by sequential adding edges between non-adjacent vertices whose degree sum is at least $4n/3-4/3$. We show that this cannot be improved to less than $4n/3-5$.

Keywords: Hamilton cycles, prism of graphs, graph closures

1. INTRODUCTION

A spanning cycle in a graph is called a *Hamilton cycle*. A graph with such a cycle is called *hamiltonian*. Hamiltonian problems are one of the most studied in the graph theory, see

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surveys [7, 8]. They trace their history to Sir William Rowan Hamilton to the 1850s. Various generalizations of the concept of a Hamilton cycle were also introduced, among them, so-called k-walks and k-trees. A *k-walk* is a closed spanning walk which visits each vertex at most k times (thus a Hamilton cycle is a 1-walk) and a *k-tree* is a spanning tree with maximum degree k. It is not hard to show [9] that a graph which has a k-tree has also a k-walk and a graph which has a k-walk has a $(k+1)$-tree. Hence, the properties of "having a k-walk" and "having a k-tree" are interlaced in the following sense:

$$\text{1-walk} \Rightarrow \text{2-tree} \Rightarrow \text{2-walk} \Rightarrow \text{3-tree} \Rightarrow \text{3-walk} \cdots$$

Some sufficient and necessary conditions on a graph to have a k-walk / k-tree can be found in [5].

Recently, another property sandwiched between "having a 2-tree", i.e., a Hamilton path, and "having a 2-walk" has attracted attention of researchers [10]. This property is that the prism of a graph is hamiltonian. The *prism* of a graph G is the graph obtained from two copies of G by connecting the pairs of corresponding vertices. If G is a graph of order n and size m, then its prism has $2n$ vertices and $2m+n$ edges. We often identify one of the two copies of G in the prism with the graph G itself. It can be shown that if G has a Hamilton cycle, then its prism is hamiltonian and if its prism is hamiltonian, then G has a 2-walk [10]. Some old conjectures relaxed from "having a Hamilton cycle" to "having a hamiltonian prisms" become easy and some seem to remain still hard, e.g., it is not known whether there exists a constant k such that each k-tough graph has a hamiltonian prism (recall that a graph G is *k-tough* if, for every subset A of its vertices, $G \setminus A$ is connected or has at most $k|A|$ components). This is known to be true for the property of "having a 2-walk" [6], but the problem in the case of Hamilton cycles, originally posed by Chvátal [4], remains open for more than 25 years. Only recently, Bauer, Broersma and Veldman [1] have disproved a stronger conjecture of Chvátal that each 2-tough graph is hamiltonian by constructing a non-hamiltonian $(9/4-\epsilon)$-tough graphs.

Another concept which does not obviously translate to the case of hamiltonian prisms is the concept of graph closures. A *k-closure* of a graph G, denoted by $\mathrm{Cl}_k(G)$, is the unique graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least k until no such pair remains. See also a survey on closure concepts by Broersma, Ryjáček and Schiermeyer [3]. Thus, if G is a graph of order n, we have:

$$G = \mathrm{Cl}_{2n-3}(G) \subseteq \mathrm{Cl}_{2n-4}(G) \subseteq \ldots \subseteq \mathrm{Cl}_1(G) \subseteq \mathrm{Cl}_0(G) = K_n$$

A graph property is called k-stable if G has the property if and only if $\mathrm{Cl}_k(G)$ has. The motivation for this concept comes from the original closure of Bondy and Chvátal [2] developed for Hamilton cycles: A graph G of order n is hamiltonian if and only if $\mathrm{Cl}_n(G)$ is hamiltonian and it is known that this cannot be weakened to $\mathrm{Cl}_{n-1}(G)$, i.e., the property of "having a Hamilton cycle" is n-stable but not $(n-1)$-stable. It is also known that a property of "having a k-walk" for $k \geq 2$ is $(n-1)$-stable but not $(n-2)$-stable. We remark that a different kind of closures was developed by Ryjáček [11] for Hamilton cycles in the class of so-called claw-free graphs. All of these show a tight connection between hamiltonian problems and closures of graphs and thus the authors of [10] posed the following problem:

Problem 1. Let G be a graph of order n and let x and y be two non-adjacent vertices such that the sum of their degrees is at least n. Is it true that G has a hamiltonian prism if and only if $G + xy$ does?

In particular, this problem asks whether the property of "having a hamiltonian prism" is n-stable for graphs of order n.

In this paper, we answer this problem in negative by constructing graphs that show the property of "having a hamiltonian prism" is not k-stable for $k = 4n/3 - 16/3$ (Proposition 1). On the other hand, the main result of this paper is that the prism of a graph G of order n is hamiltonian if and only if the prism of $\mathrm{Cl}_k(G)$ is hamiltonian for $k = 4n/3 - 4/3$ (Theorem 2). It seems that this could be little improved by tedious case analysis. We think that the lower bound is tight and decided to pose this as a conjecture to stimulate research to close the (quite small) gap between the upper and the lower bound:

Conjecture 1. The property of "having a hamiltonian prism" is k-stable with $k = 4n/3 - 5$ for graphs of order n and this cannot be further improved. ∎

2. THE MAIN RESULT

In this section, we present our main result. We first show by a double counting argument (which we formulate using a discharging method) that the property is k-stable with $k = 4n/3 - 1$ (Theorem 1). Next, we improve this to $k = 4n/3 - 4/3$ by a little technical case analysis.

Theorem 1. Let G be a graph of order n. Then, G has a hamiltonian prism if and only if $\mathrm{Cl}_{4n/3-1}(G)$ has a hamiltonian prism.

Proof. Let G be a fixed graph of order n. Consider two non-adjacent vertices x and y of G such that the sum of $\deg_G(x)$ and $\deg_G(y)$ is at least $4n/3 - 1$. In order to prove the theorem, it is enough to show that the prism of G is hamiltonian if and only if the prism of $G + xy$ is hamiltonian (this follows directly from the definition of $\mathrm{Cl}_{4n/3-1}(G)$).

Clearly, if the prism of G is hamiltonian, then the prism of $G + xy$ is also hamiltonian. Assume now that the prism of $G + xy$ is hamiltonian. In order to show that the prism of G is also hamiltonian, we use a double counting argument which is formulated using the discharging method.

Let us fix a Hamilton cycle C in the prism of $G + xy$ which uses the two copies of the edge xy as few times as possible. Let V and V' be the vertex sets of the copies of G. If the Hamilton cycle C omits a counterpart of the edge xy in both copies, then C is also a Hamilton cycle in the prism of G and we are done. Hence assume by way of contradiction that the cycle C traverses the image of the edge xy in the copy of G with the vertex set V. Note that it can also traverse the image of xy in the copy with the vertex set V'. Let $v_1, \ldots, v_n$ be vertices of V in the order visited by the cycle C where $v_1 = x$ and $v_n = y$. Note that some pairs v_iv_{i+1} are not edges of the cycle C; such pairs v_iv_{i+1} are further called *virtual edges*. Let v_i' be the counterpart of the vertex v_i among the vertices of V'.

A vertex v_i is said to be *vertical* if the edge v_iv_i' is contained in the cycle C. We classify the edges v_iv_{i+1} which are not virtual into three types I, II and III: An edge v_iv_{i+1} is of type I, if neither v_i nor v_{i+1} is vertical. It is of type II if exactly one of the vertices v_i and v_{i+1} is vertical. And it is of type III, if both vertices v_i and v_{i+1} are vertical. Similarly, the edge $v_nv_1 = yx$ is classified to be one of these three types. Let m_{I}, m_{II} and m_{III} be the number of edges of type I, II and III, respectively, and let n_{vert} be the number of vertical vertices. Since both ends of an virtual edge must be vertical vertices and each vertical vertex is an end of a single virtual edge, the number of virtual edges is $n_{\mathrm{vert}}/2 = m_{\mathrm{II}}/2 + m_{\mathrm{III}}$. Since each pair $v_iv_{i+1}, 1 \le i \le n-1$ (and the edge v_nv_1) is either a virtual edge or one of the types I, II and III, we have:

$$n = (m_{\mathrm{II}}/2 + m_{\mathrm{III}}) + m_{\mathrm{I}} + m_{\mathrm{II}} + m_{\mathrm{III}} = m_{\mathrm{I}} + 3m_{\mathrm{II}}/2 + 2m_{\mathrm{III}} \ . \qquad (1)$$

We now describe the discharging process. At the beginning, each edge $v_iv_{i+1}, 1 \le i \le n-1$ which is not virtual receives a charge of 1, 2 or 2 units according to whether it is of type I, II or III, respectively. The edge $v_nv_1 = yx$ does not receive any charge. Now the charge will be reassigned from edges v_iv_{i+1} to edges incident with vertices v_1 and v_n using the following rules (no charge will be reassigned to the edge $v_nv_1 = yx$). If a target edge v_1v_i or v_nv_i described in one of the following rules does not exist, the rule does not apply.

Rule R1: An edge v_1v_i receives a charge of 1 unit from the edge $v_{i-1}v_i$ if the edge $v_{i-1}v_i$ is not virtual.

Rule R2: An edge v_1v_i receives a charge of 1 unit from the edge v_iv_{i+1}, if the edge $v_{i-1}v_i$ is virtual.

Rule R3: An edge v_nv_i receives a charge of 1 unit from the edge v_iv_{i+1} if the edge v_iv_{i+1} is not virtual.

Rule R4: An edge v_nv_i receives a charge of 1 unit from the edge $v_{i-1}v_i$ if the edge v_iv_{i+1} is virtual.

Observe that if the edge $v_{i-1}v_i$ is virtual, then the edge v_iv_{i+1} is not virtual. Similarly, if the edge v_iv_{i+1} is virtual, then the edge $v_{i-1}v_i$ is not virtual. Thus, each edge v_1v_i and v_nv_i, $2 \le i \le n-1$, receives charge of exactly 1 unit since exactly one of the rules apply to it. Note that the edges v_1v_2 and v_nv_{n-1} (if they exist) receive some charge from themselves by Rules R1 and R3.

We now show each edge v_iv_{i+1} sends out at most the amount of charge that it was initially assigned. Assume the opposite and fix an edge v_iv_{i+1} which sends more. Three cases need to be considered according to the type of the edge v_iv_{i+1}:

The edge v_iv_{i+1} is of type I: The initial charge of v_iv_{i+1} is one. Since neither v_i nor v_{i+1} is vertical, the edge v_iv_{i+1} can send out some charge only using the Rules R1 and R3. Thus, if v_iv_{i+1} sends out more than the initial amount of charge, there must exist both the edges v_1v_{i+1} and v_nv_i. Consider now the cycle C' obtained from C by replacing edges v_nv_1 and v_iv_{i+1} with edges v_1v_{i+1} and v_nv_i, respectively. The cycle C' is a Hamilton cycle in the prism of $G+xy$ which uses fewer copies of xy than C—contradiction.

The edge v_iv_{i+1} is of type II: The initial charge of v_iv_{i+1} is two. Since exactly one of v_i and v_{i+1} is vertical, at most one of the Rules R2 and R4 can be applied. Thus, if v_iv_{i+1} sends out more than the initial amount of charge, both the Rules R1 and R3 apply and there exist both edges v_1v_{i+1} and v_nv_i. Similarly as in the previous case, the cycle C' obtained from C by replacing edges v_nv_1 and v_iv_{i+1} with edges v_1v_{i+1} and v_nv_i is a Hamilton cycle which uses less copies of xy than C—contradiction.

The edge v_iv_{i+1} is of type III: The initial charge of v_iv_{i+1} is two. If v_iv_{i+1} sends out more than the initial amount of charge, there must be at least three of the edges v_1v_i, v_1v_{i+1}, v_nv_i and v_nv_{i+1} present in the graph. Hence, there is definitely the pair of edges v_1v_{i+1} and v_nv_i **or** the pair of edges v_1v_i and v_nv_{i+1}. In the former case, it is possible to obtain Hamilton cycle C' in the prism of $G+xy$ which uses less copies of xy similarly as in the two previous cases. Let us now analyze the latter case. Since v_iv_{i+1} is of type III, the path $v'_iv_iv_{i+1}v'_{i+1}$ is contained in the cycle C. Consider now the cycle C'' obtained from C by replacing the path $v'_iv_iv_{i+1}v'_{i+1}$ by the edge $v'_iv'_{i+1}$ and the edge v_nv_1 by the path $v_nv_{i+1}v_iv_1$. Again, C'' is a Hamilton cycle in the prism of $G+xy$ which uses less copies of xy than C—contradiction.

The initial charge of all the edges v_iv_{i+1} is at most $m_{\mathrm{I}}+2m_{\mathrm{II}}+2m_{\mathrm{III}}-1$; the one is subtracted because the (non-virtual) edge v_nv_1 has zero initial charge. Note that if v_nv_1 is of type II or III, it is possible to subtract two instead of one. A simple calculation (depending on the type of the edge v_nv_1) using (1) yields that the initial charge is at most $4(n-1)/3$. Indeed, if v_nv_1 is of type I, then

$$m_{\mathrm{I}}+2m_{\mathrm{II}}+2m_{\mathrm{III}}-1 \le 4/3\cdot(m_{\mathrm{I}}+3m_{\mathrm{II}}/2+2m_{\mathrm{III}})-m_{\mathrm{I}}/3-1 \le 4n/3-4/3\ .$$

If v_nv_1 is of type II or III, we have:

$$m_{\mathrm{I}}+2m_{\mathrm{II}}+2m_{\mathrm{III}}-2 \le 4/3\cdot(m_{\mathrm{I}}+3m_{\mathrm{II}}/2+2m_{\mathrm{III}})-2 = 4n/3-2\ .$$

Since each edge v_1v_i and v_nv_i (including the edges v_1v_2 and $v_{n-1}v_n$) received charge of exactly one unit, we have that $\deg_G(v_1)+\deg_G(v_n)\le 4(n-1)/3$. This contradicts the assumption that $\deg_G(x)+\deg_G(y)=\deg_G(v_1)+\deg_G(v_n)\ge 4n/3-1$ and thus the prism of G is hamiltonian. ∎

Note that the initial charge in the proof of Theorem 1 can be equal to $4(n-1)/3$ only if the edge v_nv_1 is of type I and all the other non-virtual edges are of type II. Using this, we can further improve the bound of Theorem 1:

Theorem 2. Let G be a graph of order n. Then, G has a hamiltonian prism if and only if $\mathrm{Cl}_{4n/3-4/3}(G)$ has a hamiltonian prism.

Proof. Let us keep the notation of Theorem 1. It is enough to show that the equality $\deg_G(x)+\deg_G(y) = \deg_G(v_1)+\deg_G(v_n) = 4(n-1)/3$ cannot hold under the assumption that G does not have a hamiltonian prism. Let us again have a look at the analysis of the discharging process. The initial charge is $4(n-1)/3$ only if neither v_1 nor v_n is vertical and all the non-virtual edges v_iv_{i+1}, $1 \leq i \leq n-1$, are of type II. Thus, the vertices $v_2, v_3, v_5, v_6, v_8, v_9, \dots, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}$ are vertical. In addition, each edge v_iv_{i+1} must send out charge of 2 units.

Let now B denote the set of vertices in V which are vertical and let $A = V \setminus B$. Let A' and B' be the counterparts of vertices in A and B in V', respectively. Note that $v_1, v_n \in A$ by the assumption. Since all the non-virtual edges v_iv_{i+1} are of type II, each vertex v_i in A except for v_1 and v_n has its two neighbors in the cycle C among the vertices of B. Also, the vertices v_1 and v_n have a single neighbor from B in the cycle C. Thus $C[A]$ is a graph consisting of a single edge v_1v_n and $\frac{n-4}{3}$ isolated vertices. An easy degree counting argument yields that $C[A']$ also contains at least one edge. Let v'_i and v'_j be the end vertices of an edge of $C[A']$ and assume w.l.o.g. that the path of the cycle C from v_1 to v_n visits first v'_i and then v'_j.

Since the edge v_1v_2 sends out 2 units of charge, both edges v_1v_2 and v_nv_2 are present in the graph. Similarly, there are also edges v_1v_{n-1} and v_nv_{n-1}. We now distinguish several cases according to the mutual position of edges v_nv_1 and $v'_iv'_j$ on the cycle C:

$i = 1$ **and** $j = n$**:** The cycle C' which is obtained from the cycle C by replacing the edge $v'_iv'_j = v'_1v'_n$ and the path $v_{n-1}v_nv_1v_2$ with the paths $v'_1v_1v_{n-1}$ and $v'_nv_nv_2$, respectively, is a Hamilton cycle in the prism of G.

$i = n$ **and** $j = 1$**:** Consider the cycle C' obtained from C by removing the edges v_nv_1 and $v'_iv'_j = v'_nv'_1$ and adding the edges $v_1v'_1$ and $v_nv'_n$ instead. The cycle C' is a Hamilton cycle in the prism of G—contradiction.

$i = 1$ **and** $j \neq n$**:** By our assumption, both the edges $v_{j-1}v_j$ and v_jv_{j+1} are of type II and they send out charge of 2 units each. Since the cycle C uses the least number of copies of the edge xy, there are not both edges v_1v_j and v_nv_{j-1} present in G. Then, G must contain an edge v_1v_{j-1} (otherwise, the edge $v_{j-1}v_j$ could send out only one unit of charge). A symmetric argument yields the existence of an edge v_nv_{j+1}. In the cases which follow, similar reasons are needed to show existence of some edges in G, but we present them in less detail for the sake of brevity.

Remove now the edges $v'_iv'_j = v'_1v'_j$ and $v_{j-1}v_j$ and the path $v_nv_1v_2$ from the cycle C. Let P be the set of the resulting three paths obtained in this way. Now, there are two possibilities: Either the path from v_1 to v_n in the cycle C first traverses the edge $v'_iv'_j = v'_1v'_j$ and then the edge $v_{j-1}v_j$, or vice versa. In both cases, the paths in P together with edges v_nv_2 and v'_jv_j and the path $v'_1v_1v_{j-1}$ form a Hamilton cycle C' in the prism of $G+xy$ that uses less copies of xy (Figure 1)—contradiction.

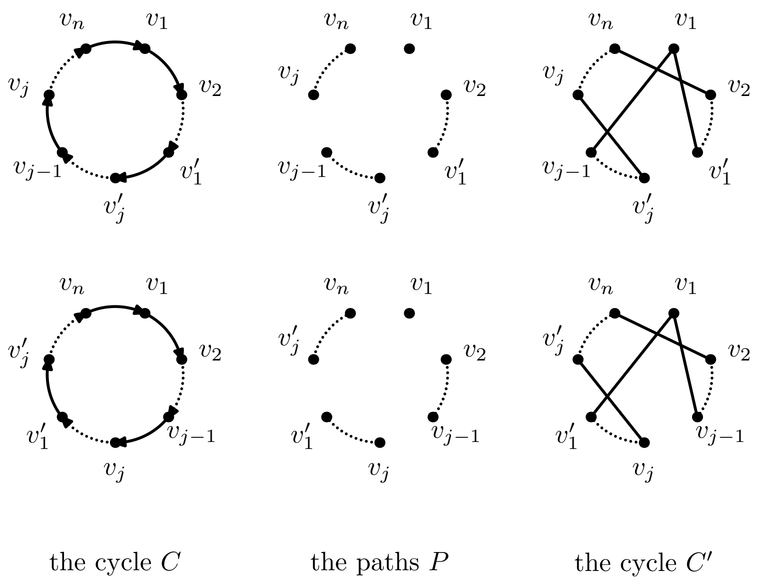

FIGURE 1. The construction of a Hamilton cycle in the proof of Theorem 2. The case $i = 1$ and $j \neq n$.

$i = n$ **and** $j \neq 1$**:** Since both the edges $v_{j-1}v_j$ and v_jv_{j+1} are of type II, they both send out charge of 2 units each and the cycle C uses the least number of copies of the edge xy, there are also edges v_1v_{j-1} and v_nv_{j+1}. Remove now the edges $v'_iv'_j = v'_nv'_j$ and $v_{j-1}v_j$ and the path $v_{n-1}v_nv_1v_2$ from the cycle C. Let P be the set of the resulting three paths obtained in this way. Now, two cases need to be considered: Either the path from v_1 to v_n in the cycle C first traverses the edge $v'_iv'_j = v'_nv'_j$ and then the edge $v_{j-1}v_j$, or vice versa. In the first case, the paths in P together with the edge $v_jv'_j$ and the paths $v_{j-1}v_1v_2$ and $v_{n-1}v_nv'_n$ form a Hamilton cycle C' in the prism of $G + xy$ that uses less copies of xy. In the other case, the paths in P together with the edge v'_jv_j and the paths $v_{j-1}v_1v_{n-1}$ and $v'_nv_nv_2$ form a Hamilton cycle C' in the prism of $G+xy$ that again uses less copies of xy. Consult also Figure 2. In both cases, this contradicts the choice of C.

$j = 1$ **and** $i \neq n$**:** This case is symmetric to the case $i = n$ and $j \neq 1$.

$j = n$ **and** $i \neq 1$**:** This case is symmetric to the case $i = 1$ and $j \neq n$.

$i, j \notin \{1, n\}$**:** Since all the edges $v_{i-1}v_i$, v_iv_{i+1}, $v_{j-1}v_j$ and v_jv_{j+1} are of type II, each of them sends out charge of 2 units and since the cycle C uses the least number of copies of the edge xy, there must also be edges v_1v_{i-1}, v_1v_{j-1}, v_nv_{i+1} and v_nv_{j+1}. Now, remove the path $v_{n-1}v_nv_1v_2$ and the edges v_iv_{i+1}, $v_{j-1}v_j$ and $v'_iv'_j$ from the cycle C and add the edges v_1v_{j-1}, v_nv_{i+1}, $v_iv'_i$ and $v_jv'_j$ instead. This operation yields two paths in the prism whose end vertices are v_1, v_2, v_{n-1} and v_n (Figure 4) for five out of six possible mutual positions of edges v_iv_{i+1}, $v_{j-1}v_j$ and $v'_iv'_j$ in the

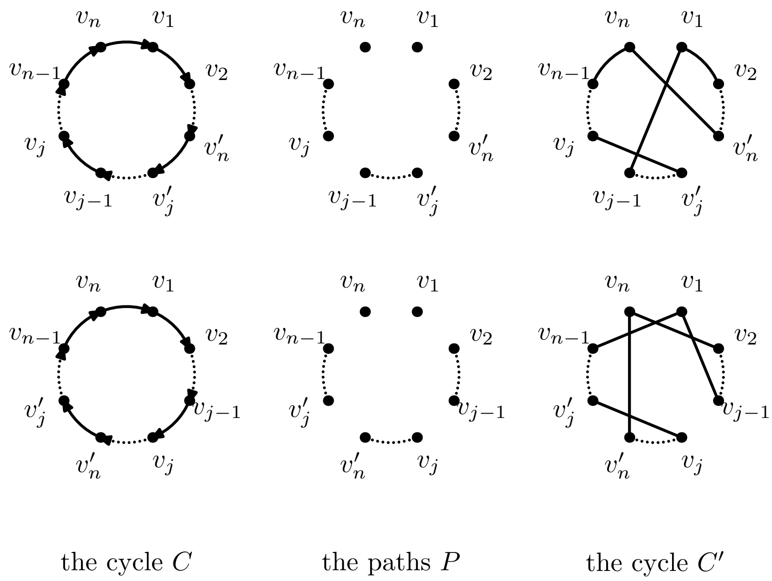

FIGURE 2. The construction of a Hamilton cycle in the proof of Theorem 2. The case $i = n$ and $j \neq 1$.

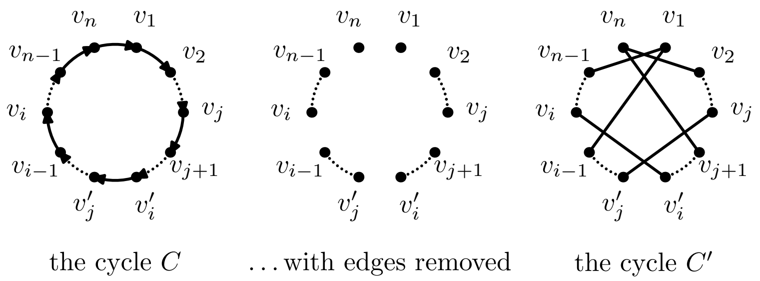

FIGURE 3. The exceptional configuration in construction of a Hamilton cycle in the proof of Theorem 2 in the case $i, j \notin \{1, n\}$.

cycle C on the path from v_1 to v_n. The exceptional configuration of the edges is the following: The cycle C first traverses the edge $v_{j-1}v_j$, then the edge $v'_i v'_j$ and then the edge $v_i v_{i+1}$. In the exceptional case, one can obtain a Hamilton cycle C' in the prism of $G + xy$ as follows: Remove the path $v_{n-1}v_n v_1 v_2$ and edges $v_{i-1}v_i$, $v_j v_{j+1}$ and $v'_i v'_j$ from C and add the edges $v_i v'_i$ and $v_j v'_j$ and the paths $v_{i-1}v_1 v_{n-1}$ and $v_2 v_n v_{j+1}$ (Figure 3). The cycle C' uses less copies of xy than C does, a contradiction.

Let us return to the general case. Since all four edges v_1v_2, v_1v_{n-1}, v_nv_2 and v_nv_{n-1} are present in G, the resulting two paths may be joined to a Hamilton cycle C' of the prism of $G + xy$ in the general case. This cycle again uses less copies of xy as C does—contradiction.

$\blacksquare$

We strongly believe that the bound on the degree sum in Theorem 2 can be further improved by a case analysis similar to that in the proof. However, the number of cases needed to consider grows quite fast and hence we decided not to follow this direction.

3. A LOWER BOUND

In this section, we show that the statement of Theorem 2 cannot be asymptotically improved:

Proposition 1. For each $k \geq 2$, there is a graph G of order $n = 3k + 4$ such that the prism of G does not have a Hamilton cycle but the prism of $\mathrm{Cl}_{4n/3-16/3}(G)$ does.

Proof. Fix an integer $k \geq 2$ and consider a complete bipartite graph $K_{k,2k}$. Let x and y be two vertices of its larger part. Identify now the vertices x and y with their counterparts in the gadget from Figure 5. Let G be the resulting graph of order $3k + 4$. The graph G for $k = 3$ is depicted in Figure 6. We show that G does not have a hamiltonian prism but $\mathrm{Cl}_{4n/3-16/3}(G)$ does.

Assume for the sake of contradiction that the prism of G has a Hamilton cycle C. Let A and B be the vertices of the smaller and the larger part of the bipartite graph $K_{k,2k}$, respectively. Note that $A \cup B$ does not contain the additional four vertices of the gadget.

We now count the number of A–B edges in each copy of G in the prism that belong to the cycle C. Since each vertex of A is isolated in $G[A]$, there is either one or two such edges incident with it in each copy of G. Hence, the numbers of A–B edges in both the copies of G are equal. On the other hand, each vertex of B except for x and y is also isolated in $G[B]$ and the cycle C can traverse the gadget only in one of the two (symmetric) ways depicted in Figure 7. Thus, the number of A–B edges in the copies of G must differ by two. This contradicts the previously established fact that they are equal. Hence, the prism of G is indeed non-hamiltonian.

Let now $v_1, \ldots, v_k$ be the vertices of A and $w_1, \ldots, w_{2k}$ the vertices of B. We can assume that $w_{2k-1} = x$ and $w_{2k} = y$. Observe that $\deg_G(v_{k-1}) + \deg_G(v_k) = 4k = 4n/3 - 16/3$. Hence, $G + v_{k-1}v_k \subseteq \mathrm{Cl}_{4n/3-16/3}(G)$. We construct a Hamilton cycle in the prism of $G + v_{k-1}v_k$. Clearly, this also establishes that $\mathrm{Cl}_{4n/3-16/3}(G)$ has a hamiltonian prism. Let v'_i be the counterpart of v_i in the other copy of G and similarly w'_i the counterpart of w_i. Consider now the following path P pasted from the segments $v_iw_{2i-1}w'_{2i-1}v'_iw'_{2i}w_{2i}v_{i+1}$ for $1 \leq i \leq k-1$. P visits each of the vertices v_i, v'_i, w_i and w'_i exactly once except for the vertices v'_k, w_{2k-1}, w'_{2k-1}, w_{2k} and w'_{2k}. Replace now in P the segment $w'_{2k-3}v'_{k-1}w'_{2k-2}$ by $w'_{2k-3}v'_{k-1}v'_kw'_{2k-2}$ and extend this new path by adding the edges $w_{2k-1}v_1$ and $w_{2k}v_k$. Let P' be the resulting path. Observe that P' contains

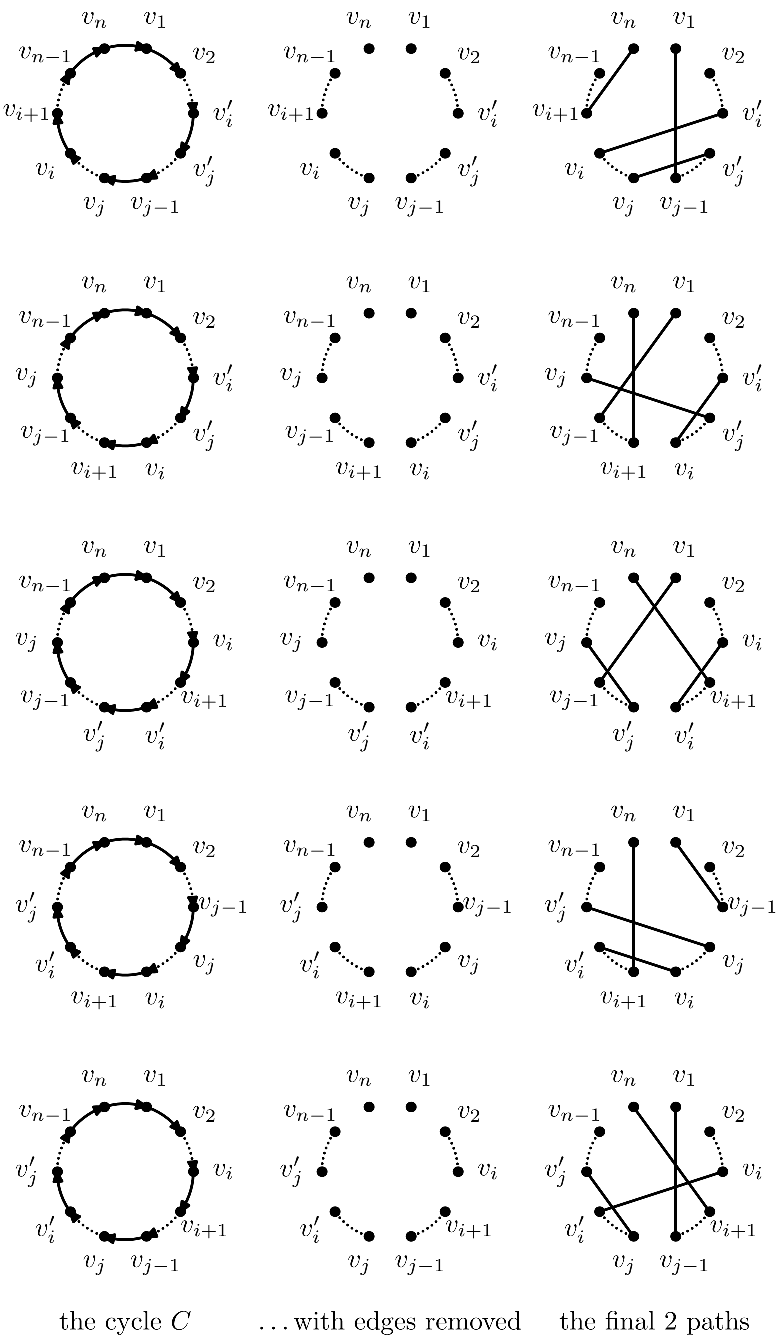

FIGURE 4. The construction of a Hamilton cycle in the proof of Theorem 2. The general case $i, j \notin \{1, n\}$.

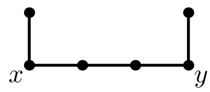

FIGURE 5. The gadget from the proof of Proposition 1.

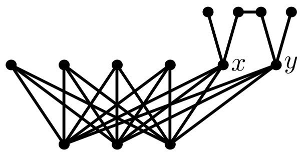

FIGURE 6. The graph G from the proof of Proposition 1 constructed for $k = 3$.

FIGURE 7. The only two possibilities how a Hamilton cycle in the prism can traverse the prism of the gadget of Figure 5.

all vertices of the prism of $K_{k,2k} + v_{k-1}v_k$, except vertices w'_{2k-1} and w'_{2k}. In addition, the end vertices of P' are $w_{2k-1} = x$ and $w_{2k} = y$. Hence P' may be extended by one of the paths depicted in Figure 7 to a Hamilton cycle in the prism of $G + v_{k-1}v_k$. ∎

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