Spanning cubic graph designs

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Abstract

Graph designs are natural extensions of BIBDs (balanced incomplete block designs). In this paper we explore spanning cubic graph designs and develop tools for constructing some of them. We show that K_{16} can be decomposed into each of the 4060 connected cubic graphs of order 16, and into precisely 144 of the 147 disconnected cubic graphs of order 16. We also identify some infinite families of cubic graphs of order $6n+4$ that decompose K_{6n+4}.

1 Introduction

We say that a graph G *decomposes* the complete graph K_n if the edges of K_n can be covered by edge-disjoint copies of G. Such a covering is then called

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a decomposition of K_n into (copies) of G. This notion was first introduced by P. Hell and A. Rosa [9], and is a natural extension of BIBDs (balanced incomplete block designs) in which blocks (complete subgraphs) are replaced by another graph G. Following BIBD notation, we use the triple $(n, G, 1)$ to indicate that the graph G decomposes K_n. If G has n vertices we call $(n, G, 1)$ a *spanning graph design*.

In this note we shall limit our discussion to spanning decompositions of K_n. The oldest spanning decomposition is probably Kirkman's [14] proof that all 1-regular graphs of order $2n$ decompose K_{2n}. This topic is still popular today; see for example the survey paper [17] and the book [25].

The well known Oberwolfach problem deals with decomposing K_n into a spanning 2-regular graph. It has only been solved for sporadic families of graphs [4,6,10,11]. When the 2-regular graph is required to be connected, i.e., Hamilton cycles, then the obvious arithmetic conditions, that is n divides the number of edges of K_n, and n is odd, are also sufficient. This follows from Walecki's famous decomposition of K_{2k+1} into k Hamilton cycles, as described by Lucas [15].

Decompositions of K_n into spanning cubic (3-regular) graphs have been considered in [3,24]. Interestingly, it is well known that the Petersen graph does not decompose K_{10} [8,23]. Consequently, most of the research concentrated on decompositions of small complete graphs into cubic graphs. W. Imrich [12] proved that there are only 21 distinct cubic graphs of order 10. P. Adams, D. Bryant and A. Khodkar [2] proved that fifteen of the 21 graphs decompose K_{10} while the other six do not. G. Khosrovshahi et al. [13] extended this work by an extensive computer search and produced a table of all possible decompositions of K_{10} into three (not necessarily isomorphic) cubic graphs. For each triple of cubic graphs G_1, G_2, G_3 they also included a count of how many non-isomorphic decompositions of K_{10} into G_1, G_2, G_3 exist. Similar results for K_{10} were also obtained by Petrenjuk [18,19].

Ringel's conjecture that every tree of order $n + 1$ decomposes K_{2n+1} [20] is probably the most famous open graph design problem. Kotzig called it the *Graph Disease*. Various labelings are powerful tools to tackle such decompositions. They were introduced by A. Rosa in [21]. His β-valuations were later renamed graceful labelings.

In this note we concentrate on cubic decompositions of complete graphs and try to extend results mentioned above to cubic graphs of arbitrary size. Clearly, if a cubic graph G forms a spanning graph design $(k, G, 1)$ then $k = 6n + 4$ for some $n \geq 1$. (The case $n = 0$ is trivial.) So our initial question was whether for each $n \geq 1$ there are cubic graphs G of order $6n + 4$ that decompose K_{6n+4}. One obvious approach was to use cyclic 1-factorizations. For each positive integer n this quickly led to cubic graphs of order $6n+4$ that decompose K_{6n+4}. Then we decided to look for cubic graphs that do not decompose K_{6n+4}. Using computer search we found that all but three cubic graphs of order 16, and all those we considered of orders 22 and 28, decomposed the corresponding complete graph. This suggests that spanning cubic graphs that do not decompose K_{6n+4} are rare.

2 Spanning cubic graph designs

In this section we develop tools to construct spanning cubic graph designs.

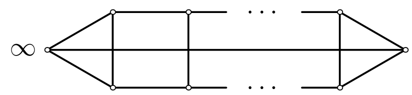

Fig. 1. A class of planar Hamiltonian cubic graphs G_{6n+4}.

As noted in [3], using well known 1-factorizations of K_{2m}, $m = 3n+2$, it is easy to construct examples of cubic graphs that decompose K_{6n+4}. For instance, the planar, Hamiltonian cubic graph in Figure 1 is obtained from the well known 1-factorization GK_{2m} (see [17]) defined by:

- $M_0 = \{(\infty, 0), (1, 2m-2), \ldots, (i, 2m-i-1), \ldots, (m-1, m)\}$
- $M_k = \{(i+k, j+k) \mid (i, j) \in M_0\},\ k = 1, \ldots, 2m-2$, with all arithmetic done mod $(2m-1)$, and $\infty + k = \infty$.

In the sequel, we use a labeling scheme that turns out to be very powerful in finding many graphs that decompose K_{6n+4}. We discuss two approaches: Breadth first search (BFS) and Depth first search (DFS). In BFS we search all cubic graphs of a fixed order. In DFS we search infinitely long sequences of cubic graphs of the same "type".

2.1 The cubic labeling, BFS

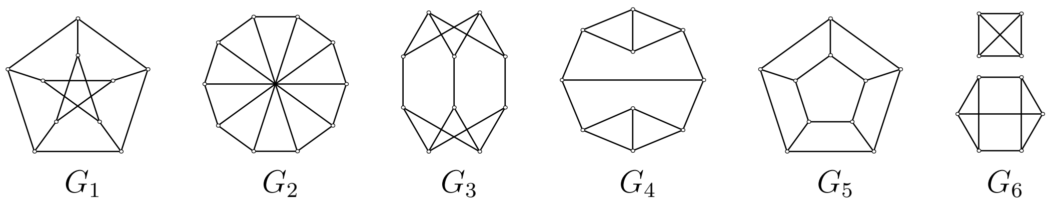

Fig. 2. The six cubic graphs of order 10 that do not decompose K_{10}.

Our first attempt was to search for some "obvious" cubic graphs of order 16 that fail to decompose K_{16}. To do so, we can start with the six cubic graphs of order 10 that fail to decompose K_{10}, c.f [2], (see Figure 2): Two of the graphs (G_2 and G_3) are bipartite and their union has chromatic number ≤ 8. The graph G_1 is the Petersen graph, which fails to decompose K_{10} for many reasons. The graph G_6 contains K_4 but its independence number is 3. The argument that the remaining two graphs fail to decompose K_{10} relies on the fact that we are trying to decompose a complete graph into three subgraphs.

First we tried to extend the six cubic graphs that failed to decompose K_{10} to cubic graphs of order 16. The first few graphs we checked included natural generalizations of these graphs and also a cubic graph of chromatic index 4; see Figures 3(a), 4 and 5.

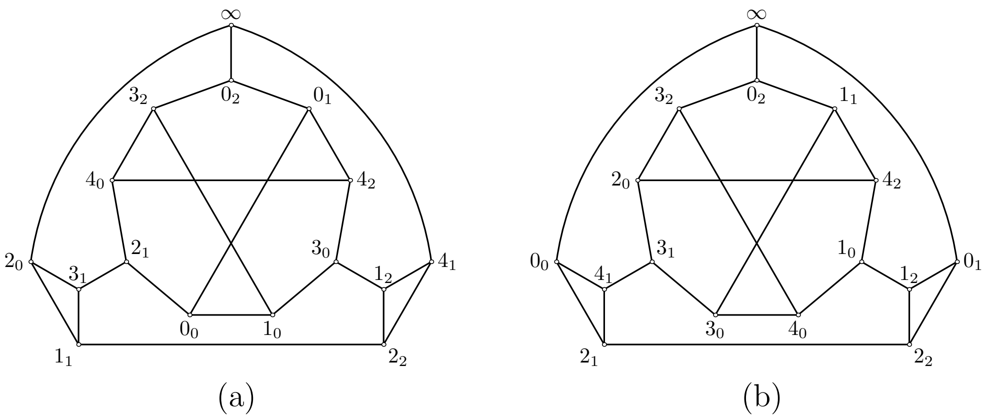

Fig. 3. Index 4 cubic graph G_1 with a cubic labeling (left), and its standard form cubic labeling (right).

For each graph G, a decomposition of K_{16} was found with a similar structure. Let the vertices of K_{16} be labeled by $\{\infty\} \cup A_0 \cup A_1 \cup A_2$ where $A_i = \{0_i, \ldots, 4_i\}$. In each case, a cyclic *starter* graph G_0 was found, with $G_0 \cong G$. The other four disjoint isomorphic copies G_j were obtained by the simple mapping $\phi_j(x_k) = (x + j \mod 5)_k$ for $j \in \{1, 2, 3, 4\}$ and $k \in \{0, 1, 2\}$, and $\phi_j(\infty) = \infty$. The graphs G_j, $0 \leq j \leq 4$, together decomposed K_{16} because the labels on G_0 were chosen such that:

- For each $0 \leq i \leq 2$, ∞ was connected to exactly one vertex in A_i, and there

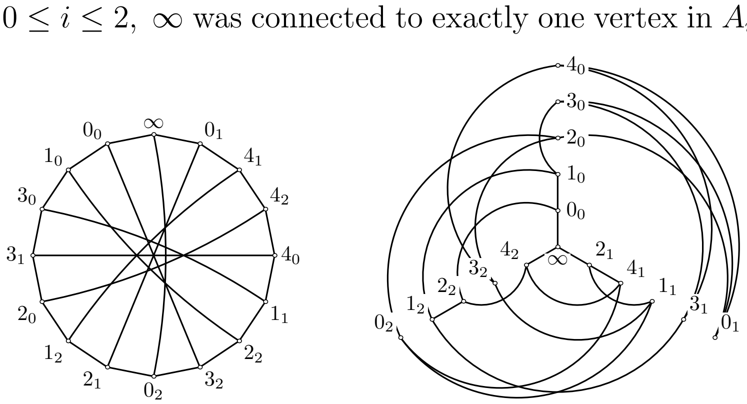

Fig. 4. The graph DG_{16} and its computer generated labeling.

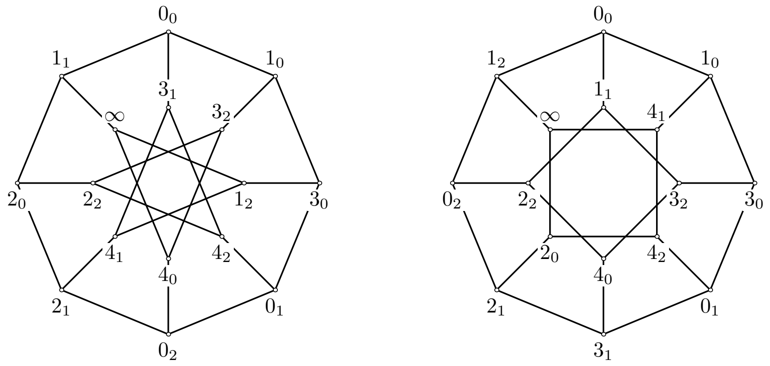

Fig. 5. Generalized Petersen graphs.

were exactly two edges between vertices spanned by each A_i.

- Exactly one of these edges has length in $\{1, 4\} \pmod 5$, and the length of the other edge is in $\{2, 3\} \pmod 5$.
- Between A_i and A_j, $j > i$ there were exactly 5 edges (x_i, y_j) such that $\{(y - x)\} \pmod 5 = \{0, 1, 2, 3, 4\}$.
- G_0 is a cubic graph.

This implied that not only did each of these graphs decompose K_{16} cyclically, but in each case there was a decomposition in which one vertex remained fixed. A computational search verified that all decompositions of the fifteen cubic graphs of order 10 also had a fixed point. This led to the following question:

> Is it true that all cubic graphs of order $6n + 4$, $n \geq 2$, decompose K_{6n+4} cyclically?

It turned out that the answer is negative, even though it is almost always true for $n = 2$: only twelve of the 4207 cubic graphs of order 16 do not decompose K_{16} cyclically. All of these are disconnected graphs. Hence we ask:

> Is it true that all connected cubic graphs of order $6n+4$, $n \geq 2$, decompose K_{6n+4} cyclically?

Among the graphs tested, and for which there exists a cyclic decomposition of K_{16}, are G_1, shown in Figure 3, the generalized Petersen graphs shown in Figure 5, $K_4 + 2PS_3$ (The symbol + represents graph union) and PS_3 is the 3-prism, and DG_{16} shown in Figure 4. A closer study of these decompositions revealed that within each A_i we may shift the labels of the vertices by a fixed amount without affecting the decomposition. This led us to the following *standard form cubic labeling*:

(i) Let $A_i = \{0_i, \ldots, (2n)_i\}$, $i = 0, 1, 2$.
(ii) Let $V(G_0) = \{\infty\} \cup A_0 \cup A_1 \cup A_2$.
(iii) Let $E(G_0) = \{(\infty, 0_0), (\infty, 0_1), (\infty, 0_2)\} \cup E_0 \cup E_1 \cup E_2 \cup E_{0,1} \cup E_{1,2} \cup E_{0,2}$.
(iv) E_i is a set of n edges (x_i, y_i) such that all differences $\{\pm(x - y) \bmod$

$(2n+1)\} = \{1, 2, \ldots 2n\}$.

(v) $E_{i,j}$ is a set of $2n+1$ edges $\{(x_i, y_j)\}$ such that $\{(x-y) \pmod{2n+1}\} = \{0, 1, \ldots, 2n\}$.

(vi) G_0 is a cubic graph.

Figure 3(a) shows the computer generated labeling of a graph, and Figure 3(b) shows the same graph relabeled in standard form.

Given the labeled graph G_0, define the graphs $G_i,\ i = 1, \ldots, 2n$ as follows:

- $V(G_i) = V(G_0)$.
- $(x_k, y_j) \in E(G_i)$ iff $((x-i)_k, (y-i)_j) \in G_0$, where arithmetic is mod $2n+1$.

It is easy to see that the mappings $\phi_i(x_k) = (x+i \bmod (2n+1))_k$ and $\phi_i(\infty) = \infty$, for $i = 1, \ldots, 2n$, map G_0 onto $2n$ isomorphic, pairwise edge-disjoint, cubic graphs on the same set of vertices, yielding the spanning graph design $(6n+4, G_0, 1)$. Following design theory custom, we call these decompositions *cyclic decompositions*.

It is clear that the freedom of choosing the edges within each E_i and $E_{i,j}$ promises a very large number of cubic graphs of order $6n+4$ that decompose K_{6n+4}. The cubic labeling in Figure 4 can be readily implemented in programs that generate spanning cubic graph decompositions. At this stage we embarked on a BFS of all cubic graphs of order 16. We independently wrote two programs that complemented each other. In the first, cubic graphs were picked from Brendan McKay's list [16] and the program tried to fit them with the cubic labeling. In the other program we started by generating all possible standard form cubic labelings and then matching these graphs with those in Brendan's list. The results were somewhat surprising. Almost all cubic graphs of order 16 cyclically decomposed K_{16}. Specifically, all 4060 connected cubic graphs of order 16 decompose K_{16} cyclically. Of the 147 disconnected cubic graphs of order 16, 135 decompose K_{16} cyclically, nine decompose K_{16} non-cyclically, and three fail to decompose K_{16}. The website [1] contains a list of all cubic graphs of order 16, with standard form cubic labelings when they exist, other decomposition descriptions and the three failed graphs. The listing of the graphs is in the same order as in Brendan McKay's list [16]. We must add that our claim that three graphs do not decompose K_{16} (namely $K_4 + K_{3,3} + PS_3$, $2K_4 + G_8$ (cf. Figure 6) and $K_{3,3}+$ the Petersen graph) are computational proofs. We could not come up with a mathematical argument to substantiate this claim.

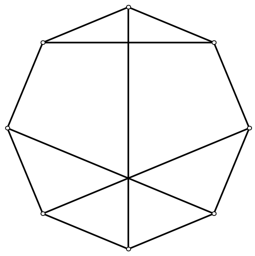

Fig. 6. The graph G_8.

2.2 The cubic labeling, DFS

In this section we use a number of approaches to demonstrate the power of the standard cubic labeling to generate infinite sequences of spanning cubic graph designs. The first such infinite sequence was constructed by Hanani et al. [7]. They constructed resolvable BIBDs $B[4,1;v]$ for all $v = 4 \mod 12$, thus proving that the cubic graph consisting of $3k+1$ K_4's decomposes K_{12k+4}. A second family was constructed by Adams et al. [3] where cubic graphs consisting of disjoint copies of the 3-cube were shown to decompose K_{16+24k}.

For our first example let T_n be the sequence of recursively defined planar cubic graphs:

- Let $T_0 = K_4$ be embedded in the plane with one vertex, say O, inside the outer triangle.
- Assume that T_{n-1} is embedded in the plane so that its outer face is a triangle.
- To construct T_n subdivide each of the three edges of the outer face of T_{n-1} by a vertex. For each of these three vertices add a new vertex (into the outer face of T_{n-1}) and join it to the corresponding vertex. Finally add three edges joining the vertices of degree one. The three added vertices can be embedded into the outer face of T_{n-1} so that the resulting graph is planar.

Clearly T_n has $6n+4$ vertices, it has one triangular face (the outer face), three 4-faces (around the center O), three 5-faces, and the remaining faces are hexagons; see Figure 7.

Theorem 2.1 *For every $n \geq 1$, T_n decomposes K_{6n+4}.*

Proof. Let $1 \leq x \leq n$ and let $x_i \in A_i$ (A_i is as defined in the standard form cubic labeling.) The following cubic labeling generates this graph:

- In each E_i select n edges: $(x_i, -x_i)$.
- In $E_{i,i+1}$ select the perfect matching $\{(x_i, (-x+1)_{i+1})\}$ (with all arithmetic done modulo $2n+1$ on the vertex labels and mod 3 on the indices).

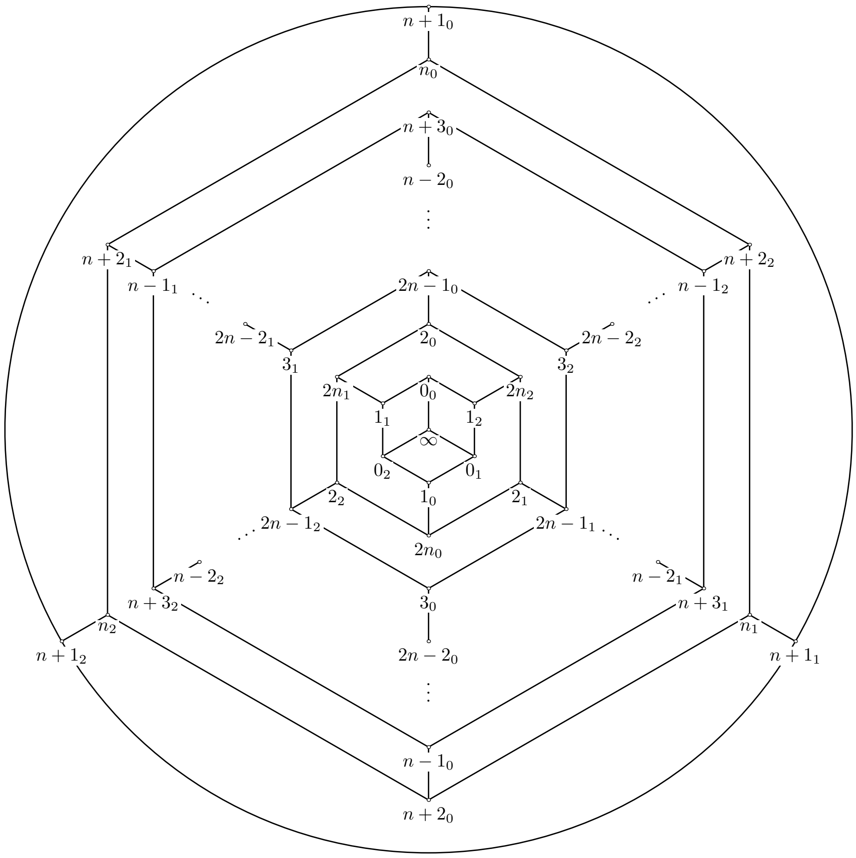

Fig. 7. The graph T_n.

Clearly, the resulting graph is a cubic graph. Since $-(n+1)+1 = -n = n+1 \pmod{2n+1}$ the vertices $\{(n+1)_0, (n+1)_1, (n+1)_2\}$ form a triangle, the outer triangle in Figure 7. The following alternative description of the graph T_n will help. T_n consists of n internal cocentric hexagons $\{C_1, \ldots, C_n\}$. $C_1 = \{0_0, 1_1, 0_2, 1_0, 0_1, 1_2\}$ and the "inward" edges $(0_k, \infty)$, $k = 0, 1, 2$. The hexagons $C_i = \{i_0, (2n+2-i)_1, i_2, (2n+2-i)_0, i_1, (2n+2-i)_2\}$, $i = 2, \ldots, n$ have "inward" edges $(i_k, (2n+1-i)_k)$, $k = 0, 1, 2$, connecting them to the hexagons C_{i-1}. The vertices of the outer triangle $\{(n+1)_0, (n+1)_1, (n+1)_2\}$ are connected by an edge to the vertices $\{n_0, n_1, n_2\}$ of the hexagon C_n. All these edges are the edges of the cubic labeling. Figure 7 provides a visual proof of the theorem. □

Another infinite sequence of spanning cubic graph designs is the sequence $K_4 + nPS_3$ (where PS_3 is the 3-prism). It is known that $K_4 + PS_3$ does not decompose K_{10}. We have:

Theorem 2.2 *For every $n \not\equiv 1 \pmod 3$, $K_4 + nPS_3$ decomposes K_{6n+4}.*

Proof. We give a short description of the applicable cubic labeling and leave the details to the reader. Let $1 \le x \le n$ and let $x_i \in A_i$ (A_i is as defined in the standard form cubic labeling.) We start with the edges $\{(x_i, -x_i)\}$ in each E_i. We add the edges $E_{0,1} = \{(x_0,(2x)_1\}$, $E_{1,2} = \{(x_1,(nx)_2)\}$ and $E_{2,0} = \{(x_2, -x_0)\}$. The set $\{\infty, 0_0, 0_1, 0_2\}$ spans a K_4. The vertices

$$\{x_0, (-x)_0, (2x)_1, (-2x)_1, (2nx)_2, (-2nx)_2\}$$

span a copy of PS_3 (note that $2nx = -x \bmod (2n+1)$). It is easy to check that this is a proper labeling and $G_0 = K_4 + nPS_3$. If $n = 3k+1$ then $\gcd(2n+1, 2n-2) = 3$ and the edges in $E_{1,2}$ will not form a cubic labeling, (all differences will be multiples of 3). □

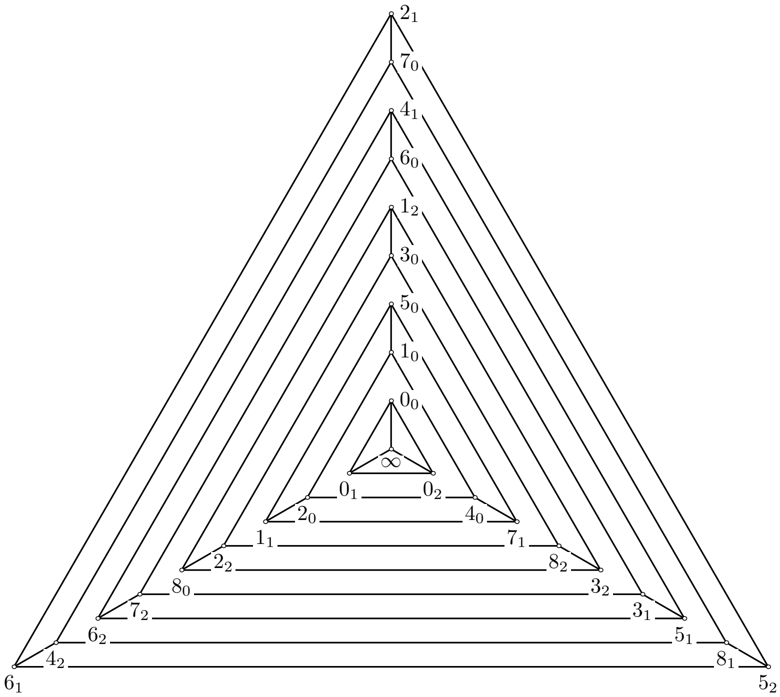

Fig. 8. The graph $K_4 + 4PS_3$.

In Figure 8 we show a standard form cubic labeling of $K_4 + 4PS_3$. We also found a standard form cubic labeling for $K_4 + 7PS_3$. These results led us to:

Conjecture 2.1 *For every $n > 1$, $K_4 + nPS_3$ decomposes K_{6n+4}.*

Here is another example of a simply stated DFS problem. Let DG_{2n} be the cubic graph C_{2n} plus the main diagonals, see Figure 4. As noted in [2,13], DG_{10} does not decompose K_{10}. On the other hand, DG_{16} and DG_{22} cyclically decompose K_{16} and K_{22} respectively. They admit a cubic labeling as in Figure 9.

We could not find a way to generalize these labelings but we conjecture:

Conjecture 2.2 *For all $n > 1$, DG_{6n+4} decomposes K_{6n+4}.*

A similar graph, $K_4 + DG_{6n}$ could be handled with a different kind of labeling,

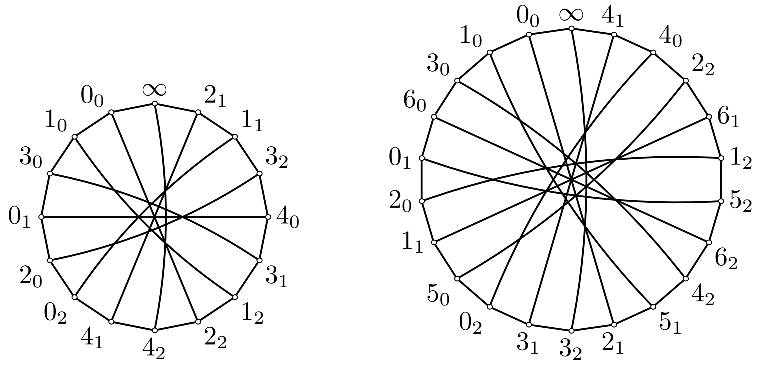

Fig. 9. DG_{16} and DG_{22} and their cubic labelings.

using group elements. The following theorem demonstrates this.

Theorem 2.3 *If* $2n+1=p^r$ *(p prime) and* $\gcd(n,3)=1$ *then* K_4+DG_{6n} *decomposes* K_{6n+4}.

Proof. Let $1 \le x \le n$ and let $x_i \in A_i$ (A_i is as defined in the standard form cubic labeling.) Let α be a primitive root in $GF(p^r)$. Label the vertices and edges of a graph G of order $6n+4$ as follows:

- The vertices $\{\infty, 0_0, 0_1, 0_2\}$ form a copy of K_4.
- $A_i = \{1_i, \alpha_i, \alpha_i^2, \ldots, \alpha_i^{2n-1}\}, i = 0, 1, 2.$
- In each A_i add a matching consisting of the edges $(x_i, (-x)_i)$.
- Add the edges $\{(x_i, (\alpha x)_{i+1})\}$, where the index is taken modulo 3.

Clearly, G is a cubic graph. Since $\alpha x - x \neq \alpha y - y$ and $\alpha^2 x - x \neq \alpha^2 y - y$, if $x \neq y$ this labeling is a standard form cubic labeling of the cubic graph G and thus it decomposes K_{6n+4}. So we need to show that $G = K_4 + DG_{6n}$. As noted above, $\{\infty, 0_0, 0_1, 0_2\}$ forms a copy of K_4. Consider the sequence:

$$S = \{1_0, \alpha_1, \alpha_2^2, \alpha_0^3, \alpha_1^4, \alpha_2^5, \ldots, \alpha_j^{2n}, \ldots, \alpha_2^{6n-1}\}.$$

We first note that the vertices in A_0 appear in this sequence in every third position, that is, as α_0^{3k}. Similarly, A_1 appears in the subsequence α_1^{3k+1} and A_2 in the subsequence α_2^{3k+2}. Also since $\gcd(n,3)=1,\ \alpha^{2n}=1,\ \alpha^{4n}=1$ will appear in the sequence with subscripts $0,1,2$. Similarly, it can be easily seen that the sequence contains all vertices in $A_0 \cup A_1 \cup A_2$. Also α_k^j is connected by an edge to α_{k+1}^{j+1}. Finally, $\alpha^{6n-1} = \alpha^{-1}$ and α_2^{-1} is connected by an edge to 1_0. Thus the sequence S forms a cycle of length $6n$. Since α is a primitive root modulo $2n+1$, $\alpha^n = -1$. For every vertex α_k^j, the vertex $\alpha_k^{3n+j} = -\alpha_k^j$ is at distance $3n$ from it on the cycle S. But these 2 vertices are connected by an edge and hence S spans a subgraph isomorphic to DG_{6n}. $\square$

Conjecture 2.3 *For every* $n \geq 1$, $DG_{6n}+K_4$ *decomposes* K_{6n+4}.

We were able to identify other sequences of spanning cubic graph designs.

These theorems demonstrated three different samples: a complete sequence, a partial but infinite sequence and using groups for labeling. Other sequences, like the cubic graphs DG_{6n+4}, are still waiting on the decomposition pile.

2.3 Concluding remarks

By [5], the following decision problem is in NP:
Input: A cubic graph G of order $6n + 4$.
Output: TRUE if G decomposes K_{6n+4}.

However, is the following problem also in NP?
Input: A cubic graph G of order $6n + 4$.
Output: TRUE if G does not decomposes K_{6n+4}.

We were not able to find "constructive" proofs or find ideas for proving that cubic graphs fail to decompose K_{6n+4}, even for a single graph of order 16.

A closely related problem is how many edge-disjoint copies of a given cubic graph G of order $2n$ can fit inside K_{2n}. It follows from a theorem of N. Sauer and J. Spencer [22] that if G has at least 18 vertices then at least two edge-disjoint copies of G can fit inside K_{2n}. We conjecture:

Conjecture 2.4 *If G is a cubic graph of order $2n$, $n \geq 7$, then G can cover at least $2/3$ of the edges of K_{2n}.*

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