Cyclic colorings of plane graphs with independent faces*

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Abstract

Let G be a plane graph with maximum face size Δ^*. If all faces of G with size four or more are vertex disjoint, then G has a cyclic coloring with $\Delta^* + 1$ colors, i.e., a coloring such that all vertices incident with the same face receive distinct colors.

1 Introduction

In 1965, Ringel [23] introduced the notion of 1-*planar graphs*. These are graphs that can be drawn in the plane such that every edge is crossed by at most one other edge. Ringel [23] proved that 1-planar graphs are 7-colorable

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and conjectured that they are 6-colorable. Ringel's conjecture was shown to be true by Borodin [5, 7] in the 1980's.

Ringel's problem fits a framework of *cyclic colorings*, vertex colorings of embedded graphs such that any two vertices incident with the same face receive distinct colors. It is easy to see that every edge-maximal 1-planar graph can be obtained from a plane graph with faces of size three and four by inserting a pair of crossing edges into each face of size four. In the other direction, removing pairs of crossing edges in an edge-maximal 1-planar graph yields a plane graph with faces of size three and four. Hence, Borodin's result [5, 7] asserts that every plane graph with maximum face size four has a cyclic coloring using at most six colors.

The holly grail in the area of cyclic colorings is the Cyclic Coloring Conjecture of Ore and Plummer [21] asserting that every plane graph with maximum face size Δ^* has a cyclic coloring with $\lfloor 3\Delta^*/2 \rfloor$ colors. The statement of the conjecture for $\Delta^* = 3$ is equivalent to the Four Color Theorem, proved in [4, 24]. For $\Delta^* = 4$, as mentioned earlied, the statement has been verified by Borodin [5, 7]. For larger values of Δ^*, the best known bound of $\lceil 5\Delta^*/3 \rceil$ has been obtained by Sanders and Zhao [25] improving earlier bounds of Borodin [6, 8]. A major evidence that the conjecture is true is a recent breakthrough of Amini, Esperet and van den Heuvel [3] which extends an approach of Havet, van den Heuvel, McDiarmid and Reed [11, 12]; Amini et al. [3] showed that the Cyclic Coloring Conjecture is asymptotically true, i.e., for every $\varepsilon > 0$, there exists Δ_ε such that every plane graph of maximum face size $\Delta^* \geq \Delta_\varepsilon$ admits a cyclic coloring with at most $\left(\frac{3}{2} + \varepsilon\right) \Delta^*$ colors.

The Cyclic Coloring Conjecture stipulated a lot of research, in particular, several restrictions and generalizations of the conjecture has been considered. Plummer and Toft [22] conjectured that the asserted bound can be improved for 3-connected plane graphs to $\Delta^* + 2$. The conjecture of Plummer and Toft is known [9, 15, 16, 17] to be true for $\Delta^* \in \{3, 4\}$ and $\Delta^* \geq 18$. In another direction, a possible generalization avoiding the restriction of face sizes, the Facial Coloring Conjecture, was proposed in [18]. This generalization asserts that vertices of every plane graph can be colored with at most $3\ell + 1$ colors in such a way that every two vertices joined by a facial walk of length at most ℓ receive distinct colors. Partial results towards proving this conjecture, which implies the Cyclic Coloring Conjecture for odd values of Δ^*, can be found in [13, 14, 18, 19].

In this paper, we consider a different restriction of the Cyclic Coloring Conjecture which is also motivated by colorings of graphs drawn in the plane with restricted structure of crossings, originally introduced by Albertson [1]. Two distinct crossings are independent if the end-vertices of every pair of crossing edges are mutually different. In particular, if all crossings are independent, then each edge is crossed by at most one edge, i.e., graphs with mutually independent crossings are 1-plane graphs. Albertson conjectured [1, 2] that every graph that can be drawn in the plane with all its crossings independent is 5-colorable and provided partial results towards the proof of his conjecture (other partial results can be found in [10, 26]). In the cyclic coloring setting, Albertson's conjecture says that every plane graph with faces of size three and four such that all faces of size four are vertex-disjoint is 5-colorable.

Albertson's conjecture has been verified by two of the authors in [20]. A natural question is what is the least number of colors needed if the maximum face size Δ^* is larger than four and the faces of size four or more are still vertex disjoint. The wheels are plane graphs of this type and thus the number of colors needed is at least $\Delta^* + 1$. We prove that this number also suffices.

2 Overview

Let us first introduce some additional notation. A vertex of degree d is a *d-vertex* and a face incident with k vertices is a *k-face*. The graphs we consider throughout the proof have no loops and no 2-faces but they can have parallel edges, in which case the degree of the vertex is considered to be the number of edges incident with it, not the number of its neighbors. Two vertices are *cyclic neighbors* if they are incident with the same face. The *cyclic degree* of a vertex v is the number of distinct cyclic neighbors of v.

A plane graph G is *D-minimal* if it has no cyclic coloring with at most $D+1$ colors, it has maximum face size at most D, all its faces of size four or more are vertex-disjoint, and G has the minimal number of vertices subject to the previous constraints. Clearly, a D-minimal graph is 2-connected and has no separating cycles of length two or three. We will use these facts implicitly throughout the paper.

Our goal is to show that there is no D-minimal graph with $D \geq 5$ (see Theorem 12). This will combine with the previous results to the following:

Theorem 1. *Every plane graph with maximum face size Δ^* whose all faces of size four or more are vertex-disjoint has a cyclic coloring with at most $\Delta^* + 1$ colors.*

The general structure of the proof is the following. We first identify configurations that cannot appear in a D-minimal graph; these configurations will be called *reducible configurations*. Using the knowledge of reducible configurations, we exclude the existence of a D-minimal graph by assigning each vertex and face charge in such a way that the total amount of charge is negative. The assigned charge is then redistributed using rules preserving its amount. The original amount of total charge will be -12 and we will be able to show that the final amount of charge of all vertices and faces is non-negative. This will exclude the existence of a D-minimal graph.

3 Reducible configurations

In this section, we study configurations that cannot appear in a D-minimal graph G. Let us start with a simple observation on the minimum degree of a D-minimal graph.

Lemma 2. *The minimum degree of every D-minimal graph G, $D \geq 5$, is at least four.*

Proof. It is straightforward to show that G has no 1-vertex. Assume that G has a d-vertex v, $d \in \{2, 3\}$. If v is incident with 3-faces only, then proceed as follows: remove v from G and consider a cyclic $(D+1)$-coloring of the resulting graph which exists by the minimality of G. This coloring can be extended to v since the cyclic degree of v is at most $3 \leq D$. Hence, we assume that v is incident with an ℓ-face, $\ell \geq 4$.

Let w and w' be the neighbors of v incident with the ℓ-face and G' the graph obtained from G by removing v and adding the edge ww' if the degree of v is three. Observe that the maximum face size of G' does not exceed the maximum face size of G and the faces of size four and more are still vertex-disjoint.

Consider a cyclic $(D+1)$-coloring of G' which exists by the minimality of G. We now construct a cyclic $(D+1)$-coloring of G. The vertices of G distinct from v preserve their colors. There are at most D colors that cannot be assigned to v: the colors of the $\ell - 1 \leq D - 1$ colors incident with the

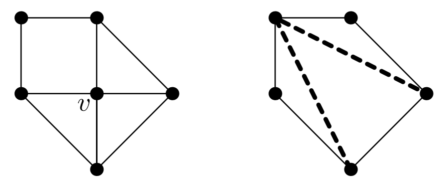

Figure 1: A reduction of a 4-vertex incident with a 4-face.

ℓ-face and the color of the third neighbor of v if v is a 3-vertex. We conclude that there is a color that can be assigned to v and thus the coloring can be completed to a cyclic $(D+1)$-coloring of G. □

In the next lemma, we look at vertices of degree four and five in D-minimal graphs.

Lemma 3. *Let G be a D-minimal graph, $D \geq 5$. Every d-vertex, $d \in \{4,5\}$ is incident with an ℓ-face, $\ell \geq 4$.*

Proof. Consider a d-vertex v, $d \in \{4,5\}$, contained only in 3-faces. Let G' be the graph obtained from G by removing v and triangulating the new d-face. By the minimality of G, the graph G' has a cyclic $(D+1)$-coloring. We now extend this coloring to G. The vertices distinct from v keep their colors. Since the cyclic degree of v in G is at most $d \leq D$, the coloring can be extended to v which contradicts our assumption that G is D-minimal. □

Next, we show that 4-vertices can be incident with 3-faces and ℓ-faces, $\ell \geq 5$, only.

Lemma 4. *Let G be a D-minimal graph, $D \geq 5$. No 4-face of G contains a 4-vertex.*

Proof. Let v be a 4-vertex incident with a 4-face f and let v' be the vertex of the 4-face not adjacent to v. By removing v from G and triangulating the resulting 5-face with edges incident with v', we obtain a graph G' (see Figure 1). By the minimality of G, the constructed graph G' has a cyclic $(D+1)$-coloring. Since the cyclic degree of v is 5 and $D \geq 5$, there is a color that can be assigned to v. This completes the coloring to a cyclic $(D+1)$-coloring of G. □

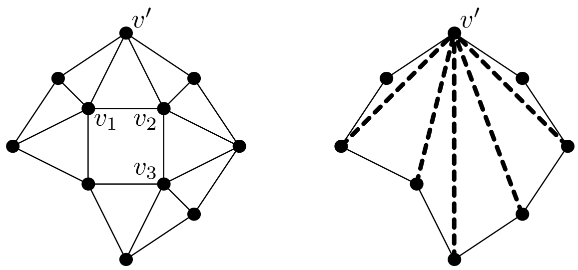

Figure 2: A reduction of a 4-face incident with three 5-vertices.

Our next goal is to exclude the cases that a 4-face or a 5-face is incident with too many 5-vertices. This is done in the next two lemmas.

Lemma 5. *Let G be a D-minimal graph, $D \geq 5$. No 4-face of G contains three 5-vertices.*

Proof. Assume that G contains a 4-face incident with three 5-vertices v_1, v_2 and v_3 (in this order on the boundary). Let v' be the common neighbor of v_1 and v_2 (see Figure 2). Remove the vertices v_1, v_2 and v_3 from G and triangulate the new 8-face with edges originating from v'. Note that the obtained graph G' has no loops since G has no separating triangles. By the minimality of G, the graph G' has a cyclic $(D+1)$-coloring.

We now extend this coloring to G. Since $D \geq 5$, there are at least 6 colors in total which can be used in the coloring. Let a be the color of v'. Color the vertex v_3 with a. We next color the vertices v_1 and v_2. Each of these two vertices has cyclic degree 6 but two of its cyclic neighbors (v_3 and v') have the same color. As $D \geq 5$, the coloring can be extended to a cyclic $(D+1)$-coloring. □

Lemma 6. *Let G be a D-minimal graph, $D \geq 5$. No 5-face f of G contains a vertex of degree five adjacent to a vertex of degree four or five.*

Proof. Let x, c, b', d and a' be the vertices of f (in this order) and assume that the vertex x is a 5-vertex and c is a 4-vertex or a 5-vertex (see Figure 3). Let b be the common neighbor of x and c, e the common neighbor of x and a' and a the remaining neighbor of x. We now modify the graph G to another graph G'. Remove the vertex x, identify the vertices a and a', and b and b'.

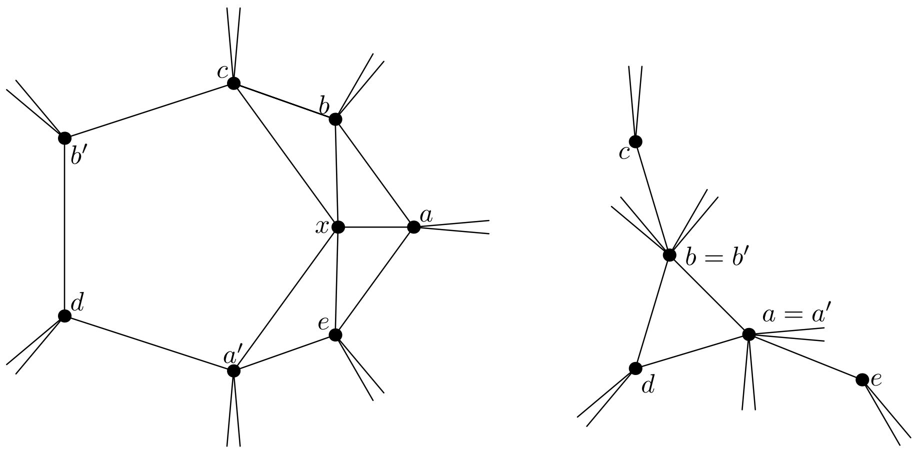

Figure 3: A 5-face incident with two consecutive 5-vertices and its reduction.

The resulting graph is G' and is loopless since G has no separating triangles. By the minimality of G, G' has a cyclic $(D+1)$-coloring.

Before extending the coloring of G' to G, we might have to recolor the vertex c (its color can coincide with the color of the vertex a' or the color of the vertex d). As the cyclic degree of c is at most $7 \leq D+2$, it has an uncolored neighbor (the vertex x) and two cyclic neighbors with the same color (the vertices b and b'), it is possible to recolor it. Finally, since the cyclic degree of x is $7 \leq D+2$ and two pairs of its cyclic neighbors (the vertices a and a', and b and b') have the same color, the coloring can be extended to x. The existence of a cyclic $(D+1)$-coloring of G contradicts the minimality of G. □

In the remaining three lemmas, we consider degrees of consecutive vertices on an ℓ-face, $\ell \geq 5$. We first exclude the existence of a face with two consecutive 4-vertices.

Lemma 7. *Let G be a D-minimal graph, $D \geq 5$. No ℓ-face f, $\ell \geq 5$, of G contains two consecutive 4-vertices.*

Proof. Let $v_1, \ldots, v_\ell$ be the vertices incident with f listed in the order on its boundary and assume that v_1 and v_2 are 4-vertices (see Figure 4). Further, let v' be the common neighbor of v_1 and v_ℓ. Form a graph G' by removing v_1 and v_2, adding the edge v_3v_ℓ and triangulating the new 5-face by adding

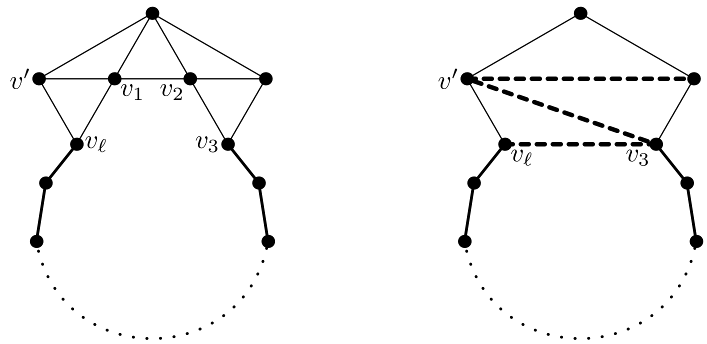

Figure 4: A reduction of a face with two consecutive 4-vertices.

edges originating from v' as in Figure 4. Since G has no separating triangles, G' is loopless. Consequently, the minimality of G implies that G' has a cyclic $(D+1)$-coloring.

Let a be the color assigned to the vertex v'. If the color a is assigned to none of the vertices $v_3, \ldots, v_\ell$, color v_2 with a. Otherwise color, v_2 with any available color (as the cyclic degree of v_2 is $\ell+1 \leq D+1$ and v_1 has no color, there is a color that can be used). Observe that two cyclic neighbors of v_1 now have the color a. Since the cyclic degree of v_1 is $\ell+1 \leq D+1$ and two of its cyclic neighbors have the same color, the coloring can be completed to a cyclic $(D+1)$-coloring of G. □

In the final two lemmas of this section, we exclude that one of three consecutive vertices on an ℓ-face, $\ell \geq 5$, would have degree four and the remaining two would have degree four or five.

Lemma 8. *Let G be a D-minimal graph, $D \geq 5$. No ℓ-face f, $\ell \geq 5$, of G contains three consecutive vertices with degrees $4, 5, 4$ or $4, 5, 5$ (in this order).*

Proof. Let $v_1, \ldots, v_\ell$ be the vertices incident with f listed in the order on its boundary and assume that v_1 is a 4-vertex, v_2 is a 5-vertex and v_3 is a d-vertex, $d \in \{4, 5\}$ (see Figures 5 and 6). Let G' be the graph obtained by removing the vertices v_1, v_2 and v_3, adding the edge v_4v_ℓ and triangulating the new face by adding edges originating from v' (see the figures) where v' is the common neighbor of v_1 and v_2. Again, G' has no loops as G has no

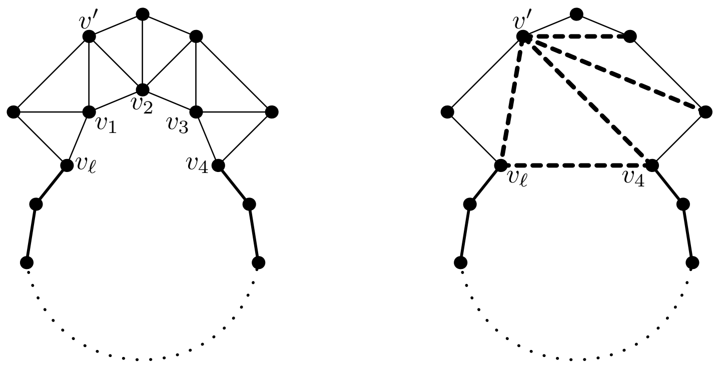

Figure 5: An ℓ-face, $\ell \geq 5$, with three consecutive vertices with degrees $4, 5, 4$ and its reduction.

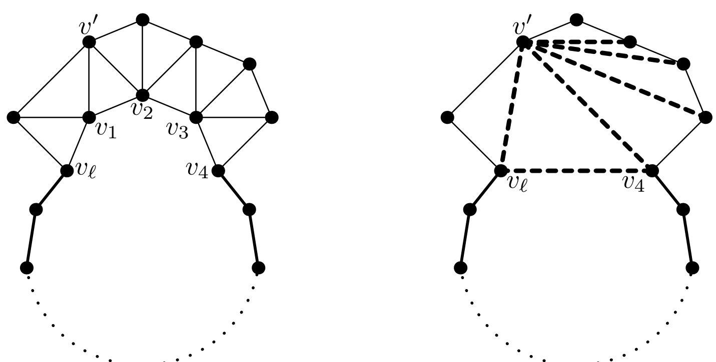

Figure 6: An ℓ-face, $\ell \geq 5$, with three consecutive vertices with degrees $4, 5, 5$ and its reduction.

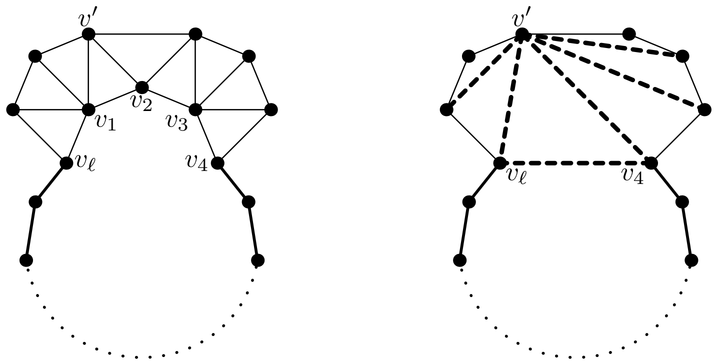

Figure 7: An ℓ-face, $\ell \geq 5$, with three consecutive vertices with degrees $5, 4, 5$ and its reduction.

separating cycles of length at most three, and the minimality of G implies that G' is cyclically $(D+1)$-colorable.

We extend a cyclic $(D+1)$-coloring of G' to G. Let a be the color assigned to the vertex v'. If the color a is not assigned to any of the vertices $v_4, \ldots, v_\ell$, assign a to v_3. Otherwise, color v_3 with any available color (as the cyclic degree of v is at most $D+2$ and two of its cyclic neighbors are uncolored, there is such an available color). Color now the vertex v_2: the cyclic degree of v_2 is $D+2$ but two of its cyclic neighbors have the same color (the color a) and one of its cyclic neighbors (the vertex v_1) is uncolored. Finally, we color the vertex v_1: since its cyclic degree is $D+1$ and two of its cyclic neighbors have the same color, there is a color that can be assigned to v_1. The existence of a cyclic $(D+1)$-coloring of G contradicts the minimality of G. $\square$

Lemma 9. *Let G be a D-minimal graph, $D \geq 5$. No ℓ-face f, $\ell \geq 5$, of G contains three consecutive vertices with degrees $5, 4, 5$ (in this order).*

Proof. The proof follows the lines of the proof of Lemma 8. We assume that G has a face f with vertices $v_1, \ldots, v_\ell$ such that v_1 and v_3 are 5-vertices and v_2 is a 4-vertex and we let v' to be the common neighbor of v_1 and v_2 (see Figure 7). Remove the vertices v_1, v_2 and v_3, add the edge $v_4 v_\ell$ and triangulate the obtained new face with edges incident with v'. The obtained graph G', which is loopless, is cyclically $(D+1)$-colorable by the minimality of G.

The cyclic $(D+1)$-coloring of G' can now be extended to G. If the color a of v' is not assigned to any of the vertices $v_4, \ldots, v_\ell$, we color v_3 with a. Otherwise, we color v_3 with any available color (as the cyclic degree of v_3 is $D+2$ and two of its cyclic neighbors are uncolored, there is an available color). We next color the vertex v_1 (its cyclic degree is $D+2$, it has an uncolored cyclic neighbor and has two cyclic neighbors colored with a) and the vertex v_2 (its cyclic degree is $D+1$ and has two cyclic neighbors colored with a). □

4 Discharging phase

In this section, we present the second part of the proof of our result. At the beginning, every d-vertex of a D-minimal graph is assigned charge of $d-6$ units and every ℓ-face is assigned charge of $2\ell-6$ units. The Euler formula implies that the total amount of charge assigned to all the vertices and faces of the graph is equal to -12. The initial charge is then redistributed based on the following two rules:

Rule 1 Every ℓ-face, $\ell \geq 4$, sends 2 units of charge to each incident 4-vertex.

Rule 2 Every ℓ-face, $\ell \geq 4$, sends 1 unit of charge to each incident 5-vertex.

First, we show that the final charge of every vertex is non-negative.

Lemma 10. *The final amount of charge of every vertex v of a D-minimal graph G, $D \geq 5$, is non-negative.*

Proof. Let d be the degree of v. By Lemma 2, the degree d is at least four. If $d \geq 6$, then the initial amount of charge of v is non-negative and its final amount of charge is also non-negative as v neither receives nor sends out any charge. If $d \in \{4, 5\}$, then v is incident with an ℓ-face, $\ell \geq 4$, by Lemma 3. If the degree of v is four, then v receive two units of charge by Rule 1, and if its degree is five, then it receive one unit of charge by Rule 2. In either of the two cases, the final amount of charge of v is zero. □

We next show that the final charge of every face is non-negative.

Lemma 11. *The final amount of charge of every face f of a D-minimal graph G, $D \geq 5$, is non-negative.*

Proof. If f is 3-face, its final amount of charge is zero. If f is a 4-face, then it is incident with no 4-vertex by Lemma 4 and with at most two 5-vertices by Lemma 5. Hence, f sends out at most two units of charge (twice one unit by Rule 2) and the final charge of f is non-negative.

Assume that f is a 5-face. By Lemma 6, f is is incident with at most two vertices of degree four or five. Consequently, Rules 1 and 2 apply at most twice and f sends out at most four units of charge to incident vertices. Since the amount of initial charge of f is equal to four units, the final charge of f is non-negative.

The remaining case is that f is an ℓ-face, $\ell \geq 6$. Let $v_1, \ldots, v_\ell$ be vertices on the boundary of f. If all vertices of f received charge from f, then they all would be 5-vertices by Lemmas 7 and 8. Hence, f would send out ℓ units of charge and its final charge would be non-negative in this case.

In what follows, we assume that f does not send out charge to all incident vertices. Let $A_1, \ldots, A_k$ be maximal consecutive intervals of vertices that receive charge from v. Observe that

$$|A_1| + \cdots + |A_k| + k \leq \ell \ . \tag{1}$$

We claim that the total amount of charge sent by f to the vertices $v_i \in A_j$ is at most $|A_j| + 1$ units for every $j = 1, \ldots, k$.

If $|A_j| = 1$, then f sends at most two units of charge to the only vertex of A_j and the claim holds. If $|A_j| = 2$, then f sends at most three units of charge to the two vertices of A_j as they both cannot be 4-vertices by Lemma 7. The claim also holds in this case. If $|A_j| \geq 3$, then none of the vertices of A_j is a 4-vertex by Lemmas 7, 8 and 9. We conclude that f sends to the vertices of A_j exactly $|A_j|$ units of charge as they all are 5-vertices.

Since the vertices of A_j receive at most $|A_j| + 1$ units of charge from f, the total amount of charge sent out by f is at most $|A_1| + \cdots + |A_k| + k$ which is at most ℓ. Since the initial amount of charge of f is $2\ell - 6$ and $\ell \geq 6$, the final amount of charge of f is non-negative. □

Lemmas 10 and 11 yield our main result:

Theorem 12. *There is no D-minimal graph, $D \geq 5$. Every plane graph with maximum face size $\Delta^* \geq 5$ whose all faces of size four or more are vertex-disjoint has a cyclic coloring with at most $\Delta^* + 1$ colors.*

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