Fault tolerant forwarding and optical indexes: a design theory approach

Arvind Gupta*   Ján Maňuch†
School of Computing Science
Simon Fraser University
Burnaby, British Columbia, Canada

Ladislav Stacho‡
Department of Mathematics
Simon Fraser University
Burnaby, British Columbia, Canada

ABSTRACT

We study the problem of designing fault tolerant routings in both complete and complete bipartite optical networks. We show that this problem has strong connections to various fundamental problems in design theory. Using a design theory approach, we find optimal f-fault tolerant arc-forwarding indexes for all complete networks and all complete balanced bipartite networks. Similarly, we find almost exact values for f-fault tolerant optical indexes for these networks.

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1. INTRODUCTION

Routing communication demands is fundamental in networking and is recognized as being of particular importance in the area of optical networking [1]. Most research concentrates on determining two basic invariants of a given optical network — the *arc-forwarding* and *optical indexes* [3]. In [10], fault tolerant issues of optical networks were considered and the two invariants were generalized into the so called *f-fault tolerant arc-forwarding index* and *f-fault tolerant optical index*. Here the parameter f represents the number of faults that are tolerated in the optical network. To determine the arc-forwarding index of a network requires the design of a path system which uses every link in the network evenly. Unfortunately, the difficulty of the problem has resulted in mainly ad hoc contructions of such systems and possible connections to other mathematical structures remain hidden.

In this paper we aim to understand the source of difficulty in path layout problems even for very simple networks. In particular, we explore connections between these layouts and well-known problems in design theory. Building on the connection with design theory, we are able to determine the f-fault tolerant arc-forwarding index and almost determine the f-fault tolerant optical index for complete networks and for complete balanced bipartite networks.

We model an all-optical network as a *symmetric* directed graph G with vertex set $V(G)$ and arc set $A(G)$, i.e., if $(u,v) \in A(G)$ then $(v,u) \in A(G)$. Let $P(u,v)$ denote a directed path in G from u to v. An *f-fault tolerant routing* in G is a set

$$\mathcal{R}_f(G) = \{P_i(u,v) : \ u, v \in V(G), \ u \neq v, \ i = 0, \ldots, f\}$$

where for each pair of distinct vertices $u, v \in V(G)$, the paths $P_0(u,v), \ldots, P_f(u,v)$ are internally vertex disjoint.

The motivation for this work, as noted in [2], is to guarantee, for a particular communication demand, fault-free transmission in a network when links and/or nodes may fail. Assuming that at most f links or nodes in a network may fail, $\mathcal{R}_f(G)$ will provide such a routing. We can view the routing $\mathcal{R}_f(G)$ as an embedding of a complete $(f+1)$-multi-digraph on n nodes into G where $|V(G)| = n$. To date, research has concentrated mainly on 0-fault tolerant routings $\mathcal{R}_0(G)$.

In practice the bandwidth of a communication link (the number of different signals a link can carry) is limited. From this perspective, it is imperative that routings be designed that minimize the maximum load on arcs. Let $\vec{\pi}(\mathcal{R}_f(G))$ denote the maximum load on arcs, that is, the maximum number of times an arc of G appears on a directed path in $\mathcal{R}_f(G)$. Then

$$\vec{\pi}_f(G) = \min_{\mathcal{R}_f(G)} \vec{\pi}(\mathcal{R}_f(G))$$

is called the *f-fault tolerant arc-forwarding index* of G. Note that the 0-fault tolerant arc-forwarding index is the well-known arc-forwarding index, cf. [11].

In an optical network, a request is serviced by sending a signal on a specific wavelength over the entire path assigned to the request by the routing. Therefore, whenever two paths of the routing share an arc, they must be assigned to different wavelengths. Let $\vec{w}(\mathcal{R}_f(G))$ be the smallest number of wavelengths that must be assigned to the directed paths of $\mathcal{R}_f(G)$ so that no two paths that share an arc receive the same wavelength.

$$\vec{w}_f(G) = \min_{\mathcal{R}_f(G)} \vec{w}(\mathcal{R}_f(G))$$

is called the *f-fault tolerant optical index* of G. Again, the 0-fault tolerant optical index is equivalent to the well-studied optical index.

Consider an f-fault tolerant routing $\mathcal{R}_f(G)$. We say that $\mathcal{R}_f$ is *optimal* if $\vec{\pi}_f(G) = \mathcal{R}_f(G)$. Furthermore, we say $\mathcal{R}_f(G)$ is *balanced* if the difference in loads between any two arcs is at most one. We are also interested in the problem of reconfiguring routings as fault tolerance needs change. For any $i = 0, \ldots, f$, *level i* of the routing $\mathcal{R}_f$ is the set of paths $P_i(u, v) \in \mathcal{R}_f$, for all $u \neq v$. It follows that for any $f' < f$, the subrouting $\mathcal{R}_{f'}(G)$ consisting of levels $0, \ldots, f'$, is f'-fault tolerant. We say that an optimal balanced routing $\mathcal{R}_f(G)$ is *leveled* if every subrouting $\mathcal{R}_{f'}(G)$ is also optimal and balanced. A feature of a leveled f-fault tolerant routing is that it can easily be reconfigured when there is a change in the requirement on the number of faults the routing should tolerate.

Our main contributions involve establishing a close connection between leveled f-fault tolerant routings in the complete digraph $\vec{K}_n$ and the existence of f disjoint idempotent Latin squares. Using this connection and tools from design theory [14, 5, 6], we find values $\vec{\pi}_f(\vec{K}_n)$ and $\vec{\pi}_f(\vec{K}_{n,n})$ for every valid combinations of f and n, and determine almost optimal values for the f-fault tolerant optical index of these networks.

This link between design theory and fault tolerance should be applicable to determining these indexes for other graphs as well as other problems in network routing. Indeed, a design theory approach was recently used in a related problem, that of minimizing the number of ADM switches in WDM ring networks (see for example [4]).

An extended abstract of this paper appeared in [9].

2. PRELIMINARIES

Let p be a path in a graph G. The *length* of p is the number of arcs of p. The *distance* of two nodes u and v of G is the length of a shortest path connecting u and v. We generalize distance as follows. The *k-distance* of u and v, $\mathrm{d}_k(u, v)$, is the sum of lengths of k distinct shortest internally node disjoint paths connecting u and v. If there are not k such paths, then we set $\mathrm{d}_k(u, v) = \infty$, and for every u, we set $\mathrm{d}_k(u, u) = 0$.

We start with the following obvious lower bound for the f-fault tolerant forwarding index.

Proposition 2.1. *For any digraph $G = (V(G), A(G))$ and $f \leq |V(G)| - 2$,*

$$\vec{\pi}_f(G) \geq \left\lceil \frac{1}{|A(G)|} \sum_{u,v \in V(G)} \mathrm{d}_{f+1}(u, v) \right\rceil.$$

To lower bound the optical index of a symmetric digraph G, consider an f-fault tolerant routing $\mathcal{R} = \mathcal{R}_f(G)$ of G. We can construct an undirected *path graph*

$\mathcal{G}_{\mathcal{R}} = G(\mathcal{R}, E(\mathcal{R}))$ as follows: two paths $P, P' \in \mathcal{R}$ are connected by an edge if P and P' share an arc. Obviously, the minimum number of wavelength needed for $\mathcal{R}$ is the same as the chromatic number of the path graph $\mathcal{G}_{\mathcal{R}}$ which must be at least $\vec{\pi}_f(G)$. Hence, we can conclude the following.

Proposition 2.2. *For any symmetric digraph G with connectivity k, and any $0 \leq f \leq k$,*

$$\vec{\pi}_f(G) \leq \vec{w}_f(G).$$

It is a well known open problem for 0-fault tolerant routings to show that $\vec{\pi}_0(G) = \vec{w}_0(G)$ for any symmetric digraph G. This is the case for many extensively studied interconnection networks and was recently also proved for symmetric trees, cf. [8]. We believe the same relation is true for f-fault tolerant routings and repeat here our conjecture from [10].

Conjecture 1. *Let G be a symmetric digraph with connectivity k. For any $0 \leq f < k$,*

$$\vec{\pi}_f(G) = \vec{w}_f(G).$$

□

3. COMPLETE DIGRAPHS

In this section, we consider complete digraphs $\vec{K}_n$ and determine their f-fault tolerant arc-forwarding and (almost determine) their f-fault tolerant optical indexes. To achieve this we establish correspondence between constructing f-fault tolerant leveled routing and designing $f+1$ disjoint idempotent Latin squares.

Consider a complete digraph $\vec{K}_n$ with $V(\vec{K}_n) = V$ ($|V| = n$) and an f-fault tolerant routing $\mathcal{R}_f(\vec{K}_n)$. Necessarily, $f \leq n-2$. For distinct $u, v \in V$, consider the $f+1$ paths in $\mathcal{R}_f(\vec{K}_n)$ connecting u and v. Since at most one of these paths has length 1, $\mathrm{d}_{f+1}(u, v) \geq 2f+1$. It follows from Proposition 2.1 that $\vec{\pi}_f(\vec{K}_n) \geq 2f+1$.

Our goal is to design leveled f-fault tolerant routings that attain this lower bound. Using some properties of Latin squares we show that for every n there is a leveled $(n-2)$-fault tolerant routing.

Definition 3.1. *A* Latin square *of order n is an $n \times n$ matrix over a set S of cardinality n such that every row (respectively column) contains each symbol of S exactly once.*

Formally, a Latin square *of order n is a pair (S, L) where L is a mapping $L : S \times S \to S$ such that for any $u, w \in S$, the equation*

$$L(u, v) = w \quad \textit{(respectively } L(v, u) = w\textit{)}$$

has a unique solution $v \in S$.

Notice that the row mapping $R_u^L : S \to S$ and the column mapping $C_u^L : S \to S$ defined as

$$R_u^L(v) = L(u,v) \quad \text{and} \quad C_u^L(v) = L(v,u) \quad \text{for all } u,v \in S,$$

are permutations.

We will follow convention by writing a Latin square of order n as an $n \times n$ matrix for which the cell in row u and column v contains the symbol $L(u,v)$.

Definition 3.2. *A Latin square* (S,L) *is* idempotent *if for every* $u \in S$, $L(u,u) = u$.

Definition 3.3. *Two idempotent Latin squares* (S,L_1) *and* (S,L_2) *are* disjoint *if for all* $u \neq v$, $L_1(u,v) \neq L_2(u,v)$.

Note that the property of disjointness of two idempotent Latin squares (S,L_1) and (S,L_2) is weaker than the property of *orthogonality* which has been studied extensively, cf. [7]. Two Latin squares (S,L_1) and (S,L_2) are orthogonal if

$$\{[L_1(u,v), L_2(u,v)] : \quad u,v \in S\} = S \times S\,.$$

Indeed, orthogonal idempotent Latin squares (S,L_1) and (S,L_2) are also disjoint, since if $L_1(u,v) = L_2(u,v) = w$ for some $u \neq v$ then

$$[L_1(u,v), L_2(u,v)] = [w,w] = [L_1(w,w), L_2(w,w)]$$

which contradicts orthogonality.

If n is a prime power, an elementary construction in [13] yields $n-2$ mutually orthogonal idempotent Latin squares of order n, and hence $n-2$ disjoint idempotent Latin squares.

Lemma 3.4 [13]. *Let* $n = p^r$ *be a prime power. Then there exist* $n-2$ *mutually orthogonal idempotent Latin squares.*

We detail this construction since it is very simple and provides an example of a routing we are looking for.

Construction 3.1. *Let* $(\mathbb{F}, \cdot, +)$ *be a finite field of order* $n = p^r$. *Then for every* $i \in \mathbb{F} - \{0,1\}$ *and* $u,v \in \mathbb{F}$ *define*

$$L_i^*(u,v) = i \cdot u + (1-i) \cdot v\,.$$

For other n except 6, the construction was given in [14] for all n except $6, 14$ and 62, in [5] for $n = 62$ and in [6] for $n = 14$.

Theorem 3.5 [14, 5, 6]. *For all* n *except* 6 *there exist* $n-2$ *(a large set of) disjoint idempotent Latin squares. For* $n = 6$ *there are only two such Latin squares.*

The next theorem shows how to use disjoint idempotent Latin squares to construct a leveled routing for a complete digraph. We say that a leveled f-fault tolerant system is *symmetric* if for every level, all paths in the level have the same length ℓ and the loads on arcs caused by the i-th arcs of those paths are the same too, for all $i = 1, .., \ell$.

Theorem 3.6. *There exists a large set of disjoint idempotent Latin squares of order n if and only if there exists a symmetric leveled $(n-2)$-fault tolerant routing for $\vec{K}_n$.*

Proof. Assume that n and f are chosen so that there exists a symmetric leveled f-fault tolerant routing $\mathcal{R}_f(\vec{K}_n)$ with load $2f+1$. Since the routing $\mathcal{R}_f(\vec{K}_n)$ is leveled, the subrouting $\mathcal{R}_0(\vec{K}_n)$ contains only paths of length 1, and each consecutive level contains only paths of length 2. Therefore the leveled routing $\mathcal{R}_f(\vec{K}_n)$ can be described using a system of functions $F_i : V \times V \to V$, $i = 1, \ldots, f$ defined as the union of the following sets:

$$\begin{aligned} \mathcal{P}_0 &= \cup_{u \neq v} \{P_0(u,v) = u \to v\}, \\ \mathcal{P}_i &= \cup_{u \neq v} \{P_i(u,v) = u \to F_i(u,v) \to v\}, \text{ for every } i = 1, \ldots, f, \end{aligned} \tag{P}$$

where the system of functions $\{F_i\}_{i=1}^f$ satisfies conditions (1), (2) and (3) described below. The notation $a_1 \to a_2 \to \cdots \to a_k$ denotes the directed path through nodes $a_1, a_2, \ldots, a_k$.

First, to ensure that the P_i's are paths of length 2, we must have for all i and for all $u \neq v$,

$$F_i(u,v) \neq u, v\,. \tag{1}$$

Next, for every $i = 1, \ldots, f$ and every pair of distinct nodes $u, v \in V$, consider all paths of level i using the arc (u,v). Since, our system is leveled, there are exactly two such paths. We can divide them into two groups: $M_i^{(1)}(u,v) = \{P_i(u,z);\ F_i(u,z) = v\}$ containing the paths which use the arc (u,v) as the first arc of the path, and $M_i^{(2)}(u,v) = \{P_i(z,v);\ F_i(z,v) = u\}$ containing the paths which use the arc (u,v) as the second arc. Since we assume that our routing is symmetric, the cardinality of both sets is one for every arc (u,v) and every $i = 1, \ldots, f$. Let us extend the definition of mappings $M_i^{(1)}$ and $M_i^{(2)}$ by requiring that the above is satisfied also for $u = v$. Hence, we have the following condition:

$$|M_i^{(1)}(u,v)| = |M_i^{(2)}(u,v)| = 1 \tag{2}$$

for all u, v and $i = 1, \ldots, f$.

Finally, we have to ensure that the paths from u to v on different levels are internally node disjoint: for every $u \neq v$ and $i \neq j$,

$$F_i(u,v) \neq F_j(u,v)\,. \tag{3}$$

A problem to find the required routing $\mathcal{R}_f(\vec{K}_n)$ is equivalent to the problem to find a system of functions $\{F_i\}_{i=1}^f$ satisfying conditions (1), (2) and (3).

Conditions (1) and (2) imply that

$$F_i(u,u) = u \tag{1'}$$

for all $u \in V$. On the other hand, conditions (1') and (2) imply (1). Hence, the problem is equivalent to constructing a system of functions $\{F_i\}_{i=1}^f$ satisfying conditions (1'), (2) and (3). Such a system of functions corresponds to a sequence of f disjoint idempotent Latin squares. This follows from the following two claims.

Claim 3.7. *A system of functions $\{F_i\}_{i=1}^f$ satisfies condition (2) if and only if (V, F_i) is a Latin square for each $i = 1, \ldots, f$.*

Proof. Fix an i. For all u, v we have the following two equalities:

$$|M_i^{(1)}(u,v)| = 1 \quad \text{iff} \quad \text{there exists exactly one } z \text{ such that } F_i(u,z) = v,$$
$$|M_i^{(2)}(u,v)| = 1 \quad \text{iff} \quad \text{there exists exactly one } z \text{ such that } F_i(z,v) = u,$$

which is true if and only if (V, F_i) is a Latin square. □

Claim 3.8. *The system of functions $\{F_i\}_{i=1}^f$ which satisfies conditions (1'), (2) and (3) corresponds to the sequence of f disjoint idempotent Latin squares.*

Proof. Obviously, for a system of functions $\{F_i\}_{i=1}^f$ which satisfies (1') we have that (S, F_i) is an idempotent Latin square, and vice versa, for every $i = 1, \ldots, f$.

Similarly, for a system of functions $\{F_i\}_{i=1}^f$ which satisfies (3), we have that the sequence of Latin squares $(S, F_1), \ldots, (S, F_f)$ is disjoint, and vice versa. □

Now, the proof of the theorem follows immediately by Claims 3.7 and 3.8. □

The above theorem shows that it is possible to construct a symmetric leveled $(n-2)$-fault tolerant routing for $\vec{K}_n$ for all n except 6. To complete the construction we will provide a (non-symmetric) leveled 4-fault tolerant routing for $\vec{K}_6$.

Theorem 3.9. *For all $n \geq 2$, we can construct a leveled $(n-2)$-fault tolerant routing for $\vec{K}_n$. In particular, $\vec{\pi}_f(\vec{K}_n) = 2f + 1$ for all n and $f \leq n-2$.*

Proof. For $n \neq 6$, the claim follows by Theorems 3.5 and 3.6. If we drop the requirement for the routing to be symmetric, then instead of condition (2), the system of functions $\{F_i\}_{i=1}^f$ has to satisfy the following condition:

$$|M_i^{(1)}(u,v)| + |M_i^{(2)}(u,v)| = 2 \qquad \text{(2')}$$

for every $u, v \in V$ and $i = 1, \ldots, f$. In other words, the number of v values the row u contains plus the number of u values the column v contains equals two.

Hence, for $n = 6$, we are looking for four functions $F_1, \ldots, F_4$ satisfying conditions (1'), (2') and (3). The following tables show one instance of functions fulfilling these conditions:

F_1	0	1	2	3	4	5
0	0	2	1	4	5	3
1	2	1	3	5	0	4
2	4	5	2	1	3	0
3	5	4	0	3	1	2
4	3	0	5	2	4	1
5	1	3	4	0	2	5

F_2	0	1	2	3	4	5
0	0	3	4	5	1	2
1	5	1	0	4	2	3
2	3	4	2	0	5	1
3	1	2	5	3	0	4
4	2	5	3	1	4	0
5	4	0	1	2	3	5

F_3	0	**1**	2	3	4	**5**
0	0	4	3	**1**	2	**1**
1	4	1	5	2	3	**0**
2	1	**3**	2	5	0	4
3	2	**5**	4	3	**5**	**0**
4	5	**3**	1	0	4	2
5	3	2	0	4	1	5

F_4	0	**1**	2	3	4	**5**
0	0	**5**	**5**	2	3	4
1	3	1	4	0	5	2
2	5	**0**	2	4	1	**3**
3	4	**0**	**1**	3	2	**1**
4	1	2	0	5	4	**3**
5	2	4	3	1	0	5

Note that this is an instance with the minimal number of asymmetries, that is the minimal number of pairs in a row or a column, depicted with bold in the above tables. Indeed, F_1 and F_2 are Latin squares, and F_3 and F_4 have each 4 asymmetries, which is the minimal number for a function satisfying conditions (1'), (2') and (3), and not being a Latin square. Recall that the maximal number of disjoint idempotent Latin squares of order 6 is two. □

3.1 An upper bound for the optical index

Consider a leveled f-fault tolerant routing $\mathcal{R}_f(\vec{K}_n) = \cup_{i=0}^{f}\mathcal{P}_i$ described in (P), where $F_1, \ldots, F_f$ are disjoint idempotent Latin squares. We will consider the subroutings $\mathcal{P}_0, \ldots, \mathcal{P}_f$ separately. For each we build the path graph and upper bound its chromatic number. Then, by assigning different sets of wavelength for the paths in different subroutings, we have the following upper bound for the wavelength number of $\mathcal{R}_f(\vec{K}_n)$:

$$\vec{w}(\mathcal{R}_f(\vec{K}_n)) \leq \sum_{i=0}^{f} \chi(\mathcal{G}_{\mathcal{P}_i})\,.$$

Lemma 3.10. *Consider a leveled f-fault tolerant routing $\cup_{i=0}^{f}\mathcal{P}_i$ described in (P), where $f \leq n-2$ and $F_1, \ldots, F_f$ are disjoint idempotent Latin squares. Then $\chi(\mathcal{G}_{\mathcal{P}_0}) = 1$, and for every $i = 1, \ldots, f$, $\chi(\mathcal{G}_{\mathcal{P}_i}) \leq 3$.*

Proof. The path graph $\mathcal{G}_{\mathcal{P}_0}$ contains no edges, hence its chromatic number is 1. Consider the path graph $\mathcal{G}_{\mathcal{P}_i}$ for some $i = 1, \ldots, f$. By (2) (respectively, by (2')), the degree of each node of $\mathcal{G}_{\mathcal{P}_i}$ is exactly 2, i.e., the path graph is a collection of cycles. Obviously, the path graph can be colored using 3 colors. □

Note that it can happen that the cycles of the path graph are of odd lengths, hence two colors are not sufficient to color the path graph $\mathcal{G}_{\mathcal{P}_i}$. We have the following upper bound for the optical index of $\vec{K}_n$.

Theorem 3.11. *Let $n = p_1^{r_1} \dots p_k^{r_k}$ where $p_1, \dots, p_k$ are distinct primes and let $p^r = \min\{p_1^{r_1}, \dots, p_k^{r_k}\}$. For every $f = 0, \dots, (p^r - 2)$, there exists a leveled f-fault tolerant routing $\mathcal{R}_f(\vec{K}_n)$ with wavelength number $\vec{w}(\mathcal{R}_f(\vec{K}_n)) \leq 3f + 1$. Consequently,*

$$2f + 1 \leq \vec{w}_f(\vec{K}_n) \leq 3f + 1.$$

4. COMPLETE BIPARTITE DIGRAPHS

Consider a complete bipartite digraph $\vec{K}_{n,n}$ with node partitioning $U = \{u_0, \dots, u_{n-1}\}$ and $V = \{v_0, \dots, v_{n-1}\}$. Let $\mathcal{R}_f(\vec{K}_{n,n})$ be an f-fault tolerant routing. Necessarily, $f \leq n - 1$.

Take a pair of distinct nodes $x, y \in U \cup V$. If x and y belong to the same partition then every path from x to y has length at least 2, and thus we have $\mathrm{d}_{f+1}(x, y) \geq 2(f + 1)$. If x and y belong to different partitions then there is only one path connecting x and y of length 1, and all other paths have length at least 3. Hence, in this case, $\mathrm{d}_{f+1}(x, y) \geq 3f + 1$. By Proposition 2.1, we have

$$\begin{aligned} \vec{\pi}_f(\vec{K}_{n,n}) &\geq \left\lceil \frac{1}{2n^2} \left(2n(n-1) \cdot 2(f+1) + 2n^2(3f+1)\right) \right\rceil \\ &= \begin{cases} 5f + 3, & \text{if } f < \frac{n}{2} - 1, \\ 5f + 2, & \text{if } \frac{n}{2} - 1 \leq f < n - 1, \\ 5f + 1, & \text{if } f = n - 1. \end{cases} \end{aligned}$$

We will show a construction of an $(n-1)$-fault tolerant leveled routing achieving this lower bound. We split the paths of the routing into the following sets:

$$\begin{aligned} \mathcal{A}_i &= \bigcup_{x \neq y} \{u_x \to v_{A_i(x,y)} \to u_y, \quad v_x \to u_{A'_i(x,y)} \to v_y\}, \quad i = 0, \dots, n-1, \\ \mathcal{B}_0 &= \bigcup_{x,y} \{u_x \to v_y, \quad v_x \to u_y\}, \\ \mathcal{B}_i &= \bigcup_{x,y} \{u_x \to v_{B_i(x,y)} \to u_{C_i(x,y)} \to v_y, \quad v_x \to u_{B'_i(x,y)} \to v_{C'_i(x,y)} \to u_y\}, \\ & \quad i = 1, \dots, n-1, \end{aligned}$$

where $A_i, A'_i, B_i, C_i, B'_i, C'_i$ are mappings $\mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$. We choose A_i, A'_i, B_i, C_i, B'_i and C'_i so that both sets $\mathcal{A} = \cup_{i \in \mathbb{Z}_n} \mathcal{A}_i$ and $\mathcal{B} = \cup_{i \in \mathbb{Z}_n} \mathcal{B}_i$ will be leveled routings for $(\vec{K}_{n,n}, U \hat{\times} U \cup V \hat{\times} V)$ and $(\vec{K}_{n,n}, U \times V \cup V \times U)$, respectively, where $S \hat{\times} S = \{(s,t) : \ s, t \in S, \ s \neq t\}$. The symbol (G, W) denotes the communication demand in which only pairs of nodes in W need to be joined by a directed path.

The following lemma describes the set of paths $\mathcal{A}$.

Lemma 4.1. *There exists a leveled $(n-1)$-fault tolerant routing $\mathcal{A} = \cup_{i\in\mathbb{Z}_n}\mathcal{A}_i$, where*

$$\mathcal{A}_i = \bigcup_{x\neq y}\{u_x \to v_{A_i(x,y)} \to u_y, \quad v_x \to u_{A'_i(x,y)} \to v_y\}, \quad i = 0,\ldots,n-1,$$

connecting nodes from the same partitions of $\vec{K}_{n,n}$ which can be constructed in quadratic time. In particular, mappings A_i and A'_i can be chosen in the following way: for all $i = 0,\ldots,n-1$,

$$\begin{aligned} A_i(x,y) &= L(-x,-y) + \phi(i) \\ A'_i(x,y) &= L'(-x,-y) + \phi(i) + 1, \text{ where} \\ \phi(i) &= \begin{cases} 2i+1 & \text{if } n \text{ is even and } i \geq \frac{n}{2}, \\ 2i & \text{otherwise,} \end{cases} \end{aligned}$$

and $(\mathbb{Z}_n, L)$ and $(\mathbb{Z}_n, L')$ are two idempotent Latin squares. Note that all arithmetic operations are taken modulo n.

A very simple construction of idempotent Latin squares of order $n \geq 3$ is presented in [12].

Proof. Since L and L' are Latin squares, for every i, the set of paths $\tilde{\mathcal{A}}_i = \bigcup_{x,y}\{u_x \to v_{L(-x,-y)+\phi(i)} \to u_y, \quad v_x \to u_{L'(-x,-y)+\phi(i)+1} \to v_y\}$ contains every arc exactly twice. Let E_i be the set of arcs of paths in $\tilde{\mathcal{A}}_i - \mathcal{A}_i$. Hence, the set E_i contains the arcs of those paths in the set $\tilde{\mathcal{A}}_i$ which start and end at the same node $x = y$, i.e.,

$$E_i = \{u_x \leftrightarrow v_{-x+\phi(i)}, \quad u_{-x+\phi(i)+1} \leftrightarrow v_x; \quad x \in \mathbb{Z}_n\},$$

The system $\cup_{i\in\mathbb{Z}_n}\mathcal{A}_i$ is leveled, if the union $E_{0,f} = \cup_{i=0}^{f} E_i$ contains every diarc

(E1) at most once, for every $f < \frac{n}{2} - 1$,
(E2) once or twice, for every $\frac{n}{2} - 1 \leq f < n - 1$, and
(E3) exactly twice, for $f = n - 1$.

Since ϕ is a permutation, the set $E_i^u = \{u_x \leftrightarrow v_{-x+\phi(i)}; \quad x, i \in \mathbb{Z}_n\}$ contains every diarc exactly once. The same is true for the set $E_i^v = \{u_{-x+\phi(i)+1} \leftrightarrow v_x; \quad x, i \in \mathbb{Z}_n\}$. Therefore, condition (E3) above is always satisfied.

To prove (E1), assume that some diarc $u_p \leftrightarrow v_q$ in $E_{0,f}$ is used twice, for some $f < \frac{1}{2} - 1$. Since ϕ is a permutation, one occurrence of $u_p \leftrightarrow v_q$ is of type $u_x \leftrightarrow v_{-x+\phi(i)}$, and the other of type $u_{-y+\phi(j)+1} \leftrightarrow v_y$. Thus, we must have

$$q = -p + \phi(i) \qquad \text{and} \qquad p = -q + \phi(j) + 1$$

for some $i, j < \frac{n}{2} - 1$. This implies $2i = \phi(i) = p + q = \phi(j) + 1 = 2j + 1$, a contradiction.

Condition (E3) implies that (E2) is equivalent to

(E2') $E_{0,\lceil \frac{n}{2} - 1\rceil}$ contains every diarc at least once.

Fix a diarc $u_p \leftrightarrow v_q$. We will show that it belongs to $E_{0,\lceil \frac{n}{2}-1\rceil}$. If $p+q \equiv 2i \bmod n$, where $i \leq \lceil \frac{n}{2} - 1\rceil$, then $u_p \leftrightarrow v_q \in E_i$ is of type $u_x \leftrightarrow v_{-x+\phi(i)}$ with $x = p$. If $p + q \equiv 2i + 1 \bmod n$, where $i \leq \lceil \frac{n}{2} - 1\rceil$, then $u_p \leftrightarrow v_q \in E_i$ is of type $u_{-x+\phi(i)+1} \leftrightarrow v_x$ with $x = q$. It follows that (E2') is satisfied.

It remains to show that the set of paths $\mathcal{A}$ is fault tolerant. Fix levels $i \neq j$, and vertices $u_x \neq u_y$. Since, ϕ is a permutation, the paths $u_x \to v_{L(-x,-y)+\phi(i)} \to u_y$ and $u_x \to v_{L(-x,-y)+\phi(j)} \to u_y$ are internally vertex disjoint. A similar argument applies to a pair of vertices $v_x \neq v_y$. □

Example: Let us illustrate the construction of Lemma 4.1 in case $n = 5$. We will consider Latin squares $(\mathbb{Z}_5, L)$ and $(\mathbb{Z}_5, L')$ to be identical, and defined as follows: $L(u, v) = L'(u, v) = 3(u+v)$. It is easy to check that they are idempotent. Hence, the mappings A_i and A'_i are defined as follows: $A_i(x, y) = 2(x + y + i)$ and $A'_i(x, y) = 2(x + y + i) + 1$.

The following diagram shows unused arcs in each level of the corresponding design:

One can see that the distributions of unused arcs is indeed nicely balanced, resulting in a leveled routing. □

The following lemma describes the set of paths $\mathcal{B}$. We will need the following definition.

Definition 4.2. *A Latin square $(\mathbb{Z}_n, A)$ is a* column-normalized, *if its columns are sorted in increasing order by the values in the first row, i.e., $A(0, j) = j$ for all $j \in \mathbb{Z}_n$.*

Lemma 4.3. *We can construct a leveled $(n-1)$-fault tolerant routing $\mathcal{B} = \cup_{i\in\mathbb{Z}_n}\mathcal{B}_i$, where*

$$\mathcal{B}_0 = \bigcup_{x,y}\{u_x \to v_y, \quad v_x \to u_x\},$$
$$\mathcal{B}_i = \bigcup_{x,y}\{u_x \to v_{B_i(x,y)} \to u_{C_i(x,y)} \to v_y, \quad v_x \to u_{B'_i(x,y)} \to v_{C'_i(x,y)} \to u_y\},$$
$$i = 1, \ldots, n-1,$$

connecting nodes from different partitions of $\vec{K}_{n,n}$ in quadratic time. In particular, mappings B_i, C_i, B'_i, C'_i can be chosen as follows:

$$B_i(x, y) = L_B(i, y), \quad C_i(x, y) = L_C(i, x),$$
$$B'_i(x, y) = L_{B'}(i, y), \quad C'_i(x, y) = L_{C'}(i, x),$$

where $(\mathbb{Z}_n, L_B)$, $(\mathbb{Z}_n, L_C)$, $(\mathbb{Z}_n, L_{B'})$, $(\mathbb{Z}_n, L_{C'})$ are any column-normalized Latin squares.

Proof. Obviously, elements of $\mathcal{B}_0$ are paths. Fix a level $i > 0$. Since, $(\mathbb{Z}_n, L_B)$, $(\mathbb{Z}_n, L_C)$, $(\mathbb{Z}_n, L_{B'})$ and $(\mathbb{Z}_n, L_{C'})$ are column-normalized Latin squares, $L_B(i, y) \neq$

y, $L_C(i,x) \neq x$, $L_{B'}(i,y) \neq y$ and $L_{C'}(i,x) \neq x$, respectively. Hence, elements of $\mathcal{B}_i$ are also paths.

Let $\mathcal{B}_i^{UV,1}, \mathcal{B}_i^{UV,2}$ and $\mathcal{B}_i^{UV,3}$ be the multisets of UV-arcs that appear as the first, second and third arc on a path in $\mathcal{B}_i$, respectively:

$$\begin{aligned}
\mathcal{B}_i^{UV,1} &= \{u_x \to v_{L_B(i,y)}; \quad x,y \in \mathbb{Z}_n\},\\
\mathcal{B}_i^{UV,2} &= \{u_{L_{B'}(i,y)} \to v_{L_{C'}(i,x)}; \quad x,y \in \mathbb{Z}_n\},\\
\mathcal{B}_i^{UV,3} &= \{u_{L_C(i,x)} \to v_y; \quad x,y \in \mathbb{Z}_n\}.
\end{aligned}$$

Since, $(\mathbb{Z}_n, L_B)$, $(\mathbb{Z}_n, L_C)$, $(\mathbb{Z}_n, L_{B'})$, $(\mathbb{Z}_n, L_{C'})$ are Latin squares, the row mappings $L_B(i,y), \ldots, L_{C'}(i,x)$ are permutations, for any fixed i. Thus, each of the multisets above contains every arc exactly once. A similar conclusion can be derived for VU-arcs. It follows that the load of every arc caused by $\mathcal{B}_i$ is 3, i.e., the set of paths $\mathcal{B}$ constitutes a leveled routing.

It remains to show that the set of paths $\mathcal{B}$ is also fault tolerant. Fix non-zero levels $i \neq j$, and vertices u_x, v_y. If the paths $u_x \to v_{L_B(i,y)} \to u_{L_C(i,x)} \to v_y$ and $u_x \to v_{L_B(j,y)} \to u_{L_C(j,x)} \to v_y$ share an internal vertex then either $L_B(i,y) = L_B(j,y)$ or $L_C(i,x) = L_C(j,x)$. Neither of those is possible since $(\mathbb{Z}_n, L_B)$ and $(\mathbb{Z}_n, L_C)$ are Latin squares. □

Lemmas 4.1 and 4.3 give the main result of this section.

Theorem 4.4. *We can construct a leveled $(n-1)$-fault tolerant routing for $\vec{K}_{n,n}$ in quadratic time. In particular, for all $f \leq n-1$,*

$$\vec{\pi}_f(\vec{K}_{n,n}) = \begin{cases} 5f+3, & \text{if } f < \frac{n}{2}-1,\\ 5f+2, & \text{if } \frac{n}{2}-1 \leq f < n-1,\\ 5f+1, & \text{if } f = n-1. \end{cases}$$

4.1 An upper bound for the optical index

Consider a leveled f-fault tolerant routing $\mathcal{R}_f(\vec{K}_{n,n})$ as described in Lemmas 4.1 and 4.3. We will consider the subroutings $\mathcal{A}_0, \ldots, \mathcal{A}_f$ and $\mathcal{B}_0, \ldots, \mathcal{B}_f$ separately. For each we build the path graph and upper bound its chromatic number. Then, by assigning different sets of wavelength for the paths in different subroutings, we have the following upper bound for the wavelength number of $\mathcal{R}_f(\vec{K}_{n,n})$:

$$\vec{w}(\mathcal{R}_f(\vec{K}_{n,n})) \leq \sum_{i=0}^{f} \chi(\mathcal{G}_{\mathcal{A}_i}) + \sum_{i=0}^{f} \chi(\mathcal{G}_{\mathcal{B}_i}).$$

For path set $\mathcal{A}_i$, we show that the corresponding path graph is bipartite. For path set $\mathcal{B}_i$ we first derive a simple upper bound on the maximum degree of the corresponding path graph. Next we show that by a special choice of Latin squares (used in construction of the system) this upper bound can be improved. Upper bounds on optical indexes will follow from Brook's theorem which gives an upper bound on chromatic number in terms of maximum degree of the graph.

Lemma 4.5. *Consider a leveled $(n-1)$-fault tolerant routing $\cup_{i \in \mathbb{Z}_n} \mathcal{A}_i$ described in Lemma 4.1. For every $i = 0, \ldots, n-1$, $\chi(\mathcal{G}_{\mathcal{A}_i}) \leq 2$.*

Proof. It is enough to prove that the path graph $\mathcal{G}_{\mathcal{A}_i}$ is a bipartite graph. We have two types of paths: UU-paths which start and end in U and VV-paths which start and end in V. It is easy to see that there are no edges between the paths of the same type. Indeed, assume for instance that for $x_1 \neq y_1$, $x_2 \neq y_2$, $(x_1, y_1) \neq (x_2, y_2)$, the paths $u_{x_1} \to v_{L(n-x_1,n-y_1)+\phi(i)} \to u_{y_1}$ and $u_{x_2} \to v_{L(n-x_2,n-y_2)+\phi(i)} \to u_{y_2}$ share the first arc. Then $x_1 = x_2 = n - x$ and $L(x, n - y_1) = L(x, n - y_2)$. Since L is a Latin square, $y_1 = y_2$, a contradiction. □

Hence, for paths in $\cup_{i=0}^{f} \mathcal{A}_i$ we need $2(f + 1)$ wavelengths. This is optimal for $f < n/2 - 1$, and differs from the optimal value by at most two for other values of f.

Lemma 4.6. *Consider a leveled $(n-1)$-fault tolerant routing $\cup_{i\in\mathbb{Z}_n}\mathcal{B}_i$ described in Lemma 4.3. Then $\chi(\mathcal{G}_{\mathcal{B}_0}) = 1$, and for every $i = 1, \ldots, n-1$, $\chi(\mathcal{G}_{\mathcal{B}_i}) \leq 6$.*

Proof. The path graph $\mathcal{G}_{\mathcal{B}_0}$ contains no edges, hence its chromatic number is 1. Consider, the graph $\mathcal{G}_{\mathcal{B}_i}$ for some $i = 1, \ldots, n-1$. We will show that the degree of each node of $\mathcal{G}_{\mathcal{B}_i}$ is at most 6. The following mappings

$$\beta(z) = L_B(i, z), \quad \gamma(z) = L_C(i, z), \quad \beta'(z) = L_{B'}(i, z) \quad \text{and} \quad \gamma'(z) = L_{C'}(i, z)$$

are permutations without a fixed point. Without loss of generality, take a UV-path $P = u_x \to v_{\beta(y)} \to u_{\gamma(x)} \to v_y \in \mathcal{B}_i$, $x \neq y$. Assume that the k-th arc of P coincides with the ℓ-th arc of another path $Q \in \mathcal{B}_i$. If $k = \ell$, then Q is also a UV-path, and since β and γ are permutations, $Q = P$, a contradiction.

If $k \neq \ell$, then there is at most one path Q such that the k-th arc of P is the ℓ-th arc of Q, since the ℓ-th arcs of such paths Q would coincide. Hence, the degree of P in $\mathcal{G}_{\mathcal{B}_i}$ is at most 6. Now, by Brook's theorem it suffices to prove that $\mathcal{G}_{\mathcal{B}_i}$ is not a complete graph. Take a $y' \neq y$, and let $P' = u_x \to v_{\beta(y')} \to u_{\gamma(x)} \to v'_y$. Since β and γ are permutations, and $x \neq \gamma(x)$, the paths P and P' do not share any arc. Hence, the graph $\mathcal{G}_{\mathcal{B}_i}$ is not complete. □

This does not give a very good upper bound for the optical index. The load induced by paths in $\mathcal{B}_i$ is 3, therefore the coloring we found might be twice the optimal. To improve the coloring we will further restrict the class of routings, by further specifying mappings B_i, C_i, B'_i, C'_i. We need the following definition.

Definition 4.7. *Let (S, L) be a Latin square. The* row-inverse *of (S, L) is a pair $(S, \bar{L})$, where $\bar{L} : S \times S \to S$ is a mapping such that $\bar{L}(u, v) = (R_u^L)^{-1}(v)$. Recall that the row mappings R_u^L are bijective.*

The next lemma checks that the row-inverse of a Latin square is also a Latin square.

Lemma 4.8. *Let (S, L) be a Latin square. The row-inverse $(S, \bar{L})$ is a Latin square. If $(\mathbb{Z}_n, L)$ is column-normalized then so is $(\mathbb{Z}_n, \bar{L})$.*

Proof. Obviously, the row mappings of $(S, \bar{L})$ are permutations. Hence, assume that for some $x, x', y \in S$, $x \neq x'$, we have $\bar{L}(x, y) = \bar{L}(x', y)$. Then $(R_x^L)^{-1}(y) =$

FIG. 1. An example of the path graph of one level of the routing $\mathcal{B}$ for complete bipartite network $\vec{K}_{4,4}$.

$(R^L_{x'})^{-1}(y) = z$. This yields

$$R^L_x(z) = R^L_x((R^L_x)^{-1}(y)) = y = R^L_{x'}(z)\,.$$

i.e., $L(x, z) = L(x', z)$, a contradiction.

The second part of the lemma follows by a simple observation that a Latin square is column-normalized if and only if the row mapping R^L_0 is the identity mapping. □

Lemma 4.9. *Consider a leveled $(n-1)$-fault tolerant routing $\cup_{i\in\mathbb{Z}_n}\mathcal{B}_i$ described in Lemma 4.3 such that $L_{B'}$ is a row-inverse of L_C and $L_{C'}$ is a row-inverse of L_B (the column-normalized Latin squares L_B and L_C can be chosen arbitrary). Then for all $i = 1,\ldots,n-1$, $\chi(\mathcal{G}_{\mathcal{B}_i}) \le 4$.*

Proof. Consider mappings $\beta, \gamma, \beta', \gamma'$ described in the proof of Lemma 4.6, and take an arbitrary path from $\mathcal{B}_i$, say $P = u_x \to v_{\beta(y)} \to u_{\gamma(x)} \to v_y$. By the additional assumption on Latin squares $L_{B'}$ and $L_{C'}$, we have $\beta' = \gamma^{-1}$ and $\gamma' = \beta^{-1}$. Consider 6 potential paths connected to P in $\mathcal{G}_{\mathcal{B}_i}$:

1. $k = 1$, $\ell = 2$, i.e., the first arc of P coincides with the second arc of Q_1. Then $Q_1 = v_{x_1} \to u_{\beta'(y_1)} \to v_{\gamma'(x_1)} \to u_{y_1}$ is a VU-path, with $x = \beta'(y_1)$ and $\beta(y) = \gamma'(x_1)$.
2. $k = 1$, $\ell = 3$. Then $Q_2 = u_{x_2} \to v_{\beta(y_2)} \to u_{\gamma(x_2)} \to v_{y_2}$, with $x = \gamma(x_2)$ and $\beta(y) = y_2$.
3. $k = 2$, $\ell = 1$. Then $Q_3 = v_{x_3} \to u_{\beta'(y_3)} \to v_{\gamma'(x_3)} \to u_{y_3}$, with $\gamma(x) = \beta'(y_3)$ and $\beta(y) = x_3$.
4. $k = 2$, $\ell = 3$. Then $Q_4 = v_{x_4} \to u_{\beta'(y_4)} \to v_{\gamma'(x_4)} \to u_{y_4}$, with $\gamma(x) = y_4$ and $\beta(y) = \gamma'(x_4)$.
5. $k = 3$, $\ell = 1$. Then $Q_5 = u_{x_5} \to v_{\beta(y_5)} \to u_{\gamma(x_5)} \to v_{y_5}$, with $\gamma(x) = x_5$ and $y = \beta(y_5)$.
6. $k = 3$, $\ell = 2$. Then $Q_6 = v_{x_6} \to u_{\beta'(y_6)} \to v_{\gamma'(x_6)} \to u_{y_6}$, with $\gamma(x) = \beta'(y_6)$ and $y = \gamma'(x_6)$.

Since $\beta' = \gamma^{-1}$ and $\gamma' = \beta^{-1}$, it is easy to observe that $Q_1 = Q_4$ and $Q_3 = Q_6$. Hence, the degree of P in $\mathcal{G}_{\mathcal{B}_i}$ is at most 4. The claim now follows by the well known Brook's theorem. □

Example: We will construct an example of the path graph $\mathcal{G}_{\mathcal{B}_i}$ for $n = 4$ and some $i > 0$. Let assume that the permutations without fixed points β, γ, β' and γ' are as follows:

$$\beta = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix} \quad \beta' = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix} \quad \gamma' = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}$$

The path graph of this level of the routing is depicted in Figure 1. Here, the paths of the type $u_x \to v_i \to u_j \to v_y$ are shown as a vertex u_{xijy}, and the paths of the type $v_x \to u_i \to v_j \to u_y$ as a vertex v_{xijy}. Note also that we indeed need four colors to color this particular path graph. □

We can now conclude an upper bound for the f-fault tolerant optical index of $\vec{K}_{n,n}$.

Theorem 4.10. *For every $f = 0, \ldots, n-1$, there exists a leveled f-fault tolerant routing $\mathcal{R}_f$ for $\vec{K}_{n,n}$ with the optical index $\vec{w}(\mathcal{R}_f) \leq 6f + 3$. Consequently,*

$$5f + 1 \leq \vec{w}_f(\vec{K}_{n,n}) \leq 6f + 3.$$

5. CONCLUSIONS

We believe the novel connection between fault tolerant routing and design theory elucidated by our results has great potential for contributions in both fields. In this paper, we used this connection to determine the fault tolerant forwarding and optical indexes of all complete and all complete balanced bipartite networks.

There is a potential to apply our techniques to broader graph classes, i.e., graphs of high connectivity or graphs that are close to being complete can be considered. More generally, problems of determing both the f-fault tolerant arc-forwarding index and the f-fault tolerant optical index for arbitrary graphs is necessary for routings in standard and optical networks.

REFERENCES

[1] A. Aggarwal, A. Bar-Noy, D. Coppersmith, R. Ramaswami, B. Schieber and M. Sudan, Efficient routing in optical networks, *J. of the ACM* **46** (1996) 973–1001.

[2] B. Beauquier, All-to-all communication for some wavelength-routed all-optical networks, *Networks* **33** (1999) 179–187.

[3] B. Beauquier, C. J. Bermond, L. Gargano, P. Hell, S. Perennes, and U. Vaccaro, Graph Problems arising from wavelength-routing in all-optical networks, in *Proc. of the 2nd Workshop on Optics and Computer Science, part of IPPS'97*, 1997.

[4] J.-C. Bermond and D. Coudert, Traffic grooming in unidirectional WDM ring networks using design theory *In ICC 2003, IEEE International Conference on Communications*, 11-15 May 2003.

[5] Y. Chang, The existence of a large set of idempotent quasigroups of order 62, *J. Combin. Math. Combin. Comput.* **32** (2000) 213–218.

[6] Y. Chang, The spectrum for large sets of idempotent quasigroups, *J. Combin. Designs* **8** (2000) 79–82.

[7] C.J. Colbourn and J.H. Dinitz, Mutually orthogonal Latin squares: a brief survey of constructions, *Journal of Statistical Planning and Inference* **95** (2001) 9–48.

[8] L. Gargano, P. Hell and S. Pérennes, Colouring paths in directed symmetric trees with applications to optical networks, *J. Graph Theory* **38** (2001) 183–186.

[9] A. Gupta, J. Maňuch and L. Stacho, Fault tolerant forwarding and optical indexes: a design theory approach, *in: Proc. of the 11th international Colloquium on Structural Information and Communication Complexity* (SIROCCO, Smolenice Castle, Slovak Republic, 2004), Springer, LNCS **3104** (2004) 197–208.

[10] J. Maňuch and L. Stacho, On f-wise arc forwarding index and wavelength allocations in faulty all-optical hypercubes, *Theor. Informatics and Appl.* **37** (2003) 255–270.

[11] Y. Manoussakis and Z. Tuza, The forwarding index of directed networks, *Discrete Appl. Math.* **68** (1996) 279–291.

[12] C.C. Lindner, C.A. Rodger, *Design theory*, CRC Press Series on Discrete Mathematics and its Applications (1997).

[13] N.S. Mendelsohn, On maximal sets of mutually orthogonal idempotent Latin squares, *Canad. Math. Bull.* **14** (1971) 449.

[14] L. Teirlinck, C.C. Lindner, The construction of large sets of idempotent quasigroups, *Europ. J. Combinatorics* **9** (1988) 83–89.