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Note

A lower bound on the number of hamiltonian cycles

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Abstract

Thomassen (J. Combin. Theory Ser. B 72 (1998) 104–109) showed that any r-regular hamiltonian graph, $r \geqslant 300$, has a second hamiltonian cycle. Refining his methods we prove: Let G be a hamiltonian graph, Δ and δ be its maximum and minimum degree, respectively. Then for any real number $k \geqslant 1$ there exists $\Delta(k)$ so that if $\Delta \geqslant \Delta(k)$ then G has at least $\delta - \lfloor \Delta/k \rfloor + 2$ hamiltonian cycles. In particular, if $k \geqslant \Delta/\delta$ and $\Delta \geqslant \Delta(k)$ then G has a second hamiltonian cycle. A simple method for calculating an upper bound on $\Delta(k)$ is given. For example, $\Delta(1) \leqslant 73$, $\Delta(1.1) \leqslant 93, \Delta(2) \leqslant 382, \Delta(50) \leqslant 545\,800$. In addition, it is shown that this bounds are nearly best possible if one confines himself to methods introduced by Thomassen.

1. Introduction

Let G be a hamiltonian graph. What conditions guarantee that G has another hamiltonian cycle? A result of Smith implies that every hamiltonian cubic graph has a second hamiltonian cycle. Thomason [4] extended the Smith's result to all graphs in which all vertices have odd degree. In particular, it means that all r-regular hamiltonian graphs, where r is odd, have a second hamiltonian cycle. Recently, Thomassen [7] proved the result for every r-regular hamiltonian graph where $r \geqslant 300$. He notes that his approach cannot be used to prove the property for 4-regular graphs, which would verify a long-standing conjecture of Sheehan [3]. By a refinement of Thomassen's methods we will prove the following theorem and discuss its limits.

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Theorem 1. *For any real number $k \geqslant 1$ there exists $\Delta(k)$ so that every hamiltonian graph G with $\Delta(G) \geqslant \Delta(k)$ has at least $\delta(G) - \lfloor \Delta(G)/k \rfloor + 2$ hamiltonian cycles. In particular, every hamiltonian graph with $\Delta(G) \geqslant \Delta(\Delta(G)/\delta(G))$ has a second hamiltonian cycle.*

Moreover, a simple method for calculating an upper bound on $\Delta(k)$ is given. For example, $\Delta(1.5) \leqslant 192$, hence, every hamiltonian graph with $\Delta(G) \geqslant 192$ has at least $\delta(G) - \frac{2}{3}\Delta(G) + 2$ hamiltonian cycles, and every r-regular graph, $r \geqslant 192$, has at least $\frac{r}{3} + 2$ hamiltonian cycles.

In [5] it is conjectured, that every hamiltonian graph of minimum degree at least 3 contains an edge e, such that both $G - e$ and G/e are hamiltonian. Clearly, existence of a second hamiltonian cycle guarantees the existence of an edge with the property. Thus, Theorem 1 provides a partial solution of the conjecture in the case $\Delta(G) \geqslant \Delta(\Delta(G)/\delta(G))$.

It can also be used for a partial answer to a conjecture stated in [8]. The conjecture relates to a problem on optimum tours in the Travelling Salesman Problem. Let C be a cycle with n vertices. Let c be any coloring of $V(C)$ in m colors, where $2m + 1 \leqslant n$. Add all edges between vertices of the same color. Then the resulting graph has a hamiltonian cycle distinct from C.

Thomassen [6] verified the conjecture in case when c is a proper coloring (i.e., no two consecutive vertices on C have the same color) and all color classes have at least three vertices. He also found a counterexample to the conjecture, hence it does not hold in general. Let m and M be the minimum and maximum number of vertices colored by the same color in the coloring c, respectively. Then Theorem 1 verifies the conjecture if there exists a real number $k \geqslant 1$ so that $M \geqslant \Delta(k)$ and $m - \lfloor M/k \rfloor \geqslant 0$.

2. Proof of theorem

Although the proof is only a refinement of Thomassen's arguments for regular graphs, for the reader's convenience, we will provide all details. Its main ingredients are the following two results.

Let H be a graph in which each edge is colored *red* or *green*. A vertex set S is called *red independent* if no two vertices of S are joined by a *red* edge. Similarly, a vertex set S is called *green dominating* if every vertex not in S is joined to a vertex in S by a *green* edge.

Lemma 1 (Thomassen [6]). *Let C be a hamiltonian cycle in a graph H. Color all edges in C by red and all edges not in C by green. If H has a vertex set which is both red independent and green dominating, then H has a second hamiltonian cycle.*

The version of Lovász' local lemma [1] used here can be found in [2, p. 79].

Lemma 2 (Lovász' Local Lemma). *Let $A_1, A_2, \ldots, A_n$, be events in a probability space. Let H be a graph with vertices $A_1, A_2, \ldots, A_n$ such that for each $i = 1, 2, \ldots, n$, A_i is independent of any combination of events that are not neighbors of A_i in G. Suppose there exist positive real numbers $x_1, \ldots, x_n$, less than* 1, *such that for each $i = 1, 2, \ldots, n$, the probability $P(A_i)$ of A_i satisfies*

$$P(A_i) < x_i \prod (1 - x_j), \tag{1}$$

where the product is taken over all j for which A_j is a neighbor of A_i. Then, $P(\bigwedge_{i=1,\ldots,n} \overline{A_i}) > 0$.

Proof of Theorem 1. The proof is based on the following statement.

Claim 1. *For any real number $k \geqslant 1$ there exists $\Delta(k)$ so that every hamiltonian graph G with $\Delta(G) \geqslant \Delta(k)$ and $\delta(G) \geqslant \lfloor \Delta(G)/k \rfloor$ has a second hamiltonian cycle.*

With this in hand, we first prove Theorem 1 and then Claim 1 will be proved.

Let $k \geqslant 1$ be a fixed number. Let G be a hamiltonian graph with the minimum degree δ and the maximum degree $\Delta \geqslant \Delta(k)$; let C be a hamiltonian cycle of G. If $\lfloor \Delta/k \rfloor > \delta$, there is nothing to prove. So suppose that $\lfloor \Delta/k \rfloor \leqslant \delta$. By Claim 1, G has a hamiltonian cycle, say C', distinct from C. Now, we remove from G an edge e belonging only to the hamiltonian cycle C' (since C and C' are distinct, this is always possible). Moreover, the edge e can be chosen in such a way that $\Delta(G - e) = \Delta(G)$. Indeed, if there are at least three such edges in C', then at least one of them cannot be incident to any fixed vertex in G (in particular, to a vertex of degree $\Delta(G)$). Similarly, if there are only two such edges, then one can observe that they cannot be incident to any common vertex, and hence at least one of them is not incident to any fixed vertex in G, again. The statement follows from the fact that C and C' always differ in at least four edges (two per cycle).

If $\delta(G - e) \geqslant \lfloor \Delta/k \rfloor$ then, by Claim 1, $G - e$ has two hamiltonian cycles and we remove another edge which belongs exactly to one of them and does not decrease the maximum degree. We repeat this process $\delta(G) - \lfloor \Delta(G)/k \rfloor + 1$ times. Now, the minimum degree δ in the resulting graph might be less than $\lfloor \Delta(G)/k \rfloor$ and we cannot repeat the above process. However, the resulting graph is hamiltonian. Therefore, in total, G has at least $\delta(G) - \lfloor \Delta(G)/k \rfloor + 2$ hamiltonian cycles.

Proof of Claim 1. Let G be a hamiltonian graph with the minimum degree δ and the maximum degree Δ; let C be a hamiltonian graph of G. Color *red* all edges of C and color *green* all other edges. We prove that G has a second hamiltonian cycle by showing that G contains a vertex set which is both red-independent and green-dominating.

Let $0 < p < 1$ be a fixed real number. Label randomly all vertices of G by labels 0 and 1, with $P(\text{label}(v) = 1) = p$ and $P(\text{label}(v) = 0) = 1 - p$ for any vertex of G. For each *red* edge e of G, let A_e be the event that the both ends of e have color 1. We will refer to these events as *edge-type* events. For each vertex v of G, let A_v

be the event that v and all the vertices of G joined to v by *green* edges have color 0. We will refer to these events as *vertex-type* events. It is not difficult to see that $\overline{A_{e_1}} \wedge \cdots \wedge \overline{A_{e_n}} \wedge \overline{A_{v_1}} \cdots \wedge \overline{A_{v_n}}$ $(n = |G|)$ is the event that G contains a set which is both red-independent and green-dominating. To prove that $P(\overline{A_{e_1}} \wedge \cdots \wedge \overline{A_{e_n}} \wedge \overline{A_{v_1}} \cdots \wedge \overline{A_{v_n}}) > 0$, we will make use of Lovász' Local Lemma.

Define a graph H as follows. Vertices of H are all events A_e and A_v. Two events of H are adjacent if the corresponding vertex sets have non-empty intersection. Clearly, each event A_i is independent of events which are not its neighbors in H.

Let $e = uv$ be any red edge of G. Now, we estimate the number of neighbors of A_e in H. Since red edges form a hamiltonian cycle in G, there are exactly two neighbors of A_e of the edge-type. In addition, there are at most $d(u)+d(v)-2 \leqslant 2\Delta - 2$ neighbors of A_e of vertex type.

Similarly, for any vertex v of G there are at most $2d(v) - 2 \leqslant 2\Delta - 2$ neighbors of A_v of the edge type in H, and at most $(d(v) - 2)(\Delta - 3) \leqslant \Delta^2 - 5\Delta + 6$ neighbors of A_v of the vertex type in H. Note that, for a graph G with girth at least 5 the above upper bounds on the number of neighbors are sharp.

Now, to each edge-type event we associate a real number $0 < x < 1$, and to each vertex-type event we associate a real number $0 < y < 1$. As $P(A_e) = p^2$ and $P(A_v) = (1-p)^{d(v)-1} \leqslant (1-p)^{\delta-1}$, one can observe that for each edge-type event condition (1) is implied by

$$p^2 < x(1-x)^2(1-y)^{2\Delta-2} \tag{2}$$

and for each vertex-type event condition (1) is implied by

$$(1-p)^{\delta-1} < y(1-x)^{2\Delta-2}(1-y)^{\Delta^2-5\Delta+6}. \tag{3}$$

Set $y = \Delta^{-2}$. Let x, c be any numbers from the interval $(0,1)$ and set $p = \sqrt{x}(1-x)c$. Then there exists $\Delta(c)$ so that (2) is satisfied for any $\Delta > \Delta(c)$ as $f(\Delta) := (1 - 1/\Delta^2)^{2\Delta-2}$ is an increasing function, and $\lim f(\Delta) = 1$ for $\Delta \to \infty$. For any number $k \geqslant 1$, we choose x so that $g(x) := [1 - \sqrt{x}(1-x)c]/[(1-x)^{2k}] < 1$. The existence of such x follows directly from the fact that $\lim g(x) = 1$ and $\lim g'(x) < 0$ for $x \to 0^+$ and any k. Now, let $k \geqslant 1$ be a fixed number. We recall that $\lfloor \Delta/k \rfloor \leqslant \delta$. Then (3) is implied by $(1-p)^{(\Delta/k)-2} < y(1-x)^{2\Delta}(1-y)^{\Delta^2}$, which in turn is equivalent to

$$\frac{\Delta^2}{(1-x)^{4k}} \left(\frac{1 - \sqrt{x}(1-x)c}{(1-x)^{2k}} \right)^{\Delta/k-2} \leqslant \left(1 - \frac{1}{\Delta^2}\right)^{\Delta^2}. \tag{4}$$

Since $\lim_{\Delta\to\infty} (\Delta^2/(1-x)^{4k}) a^{(\Delta/k)-2} = 0$ for $0 < a < 1$ and $\lim(1 - 1/\Delta^2)^{\Delta^2} = \mathrm{e}^{-1}$, there is $\Delta_1(k)$ so that (4), and consequently (3), is satisfied for all $\Delta > \Delta_1(k)$. Set $\Delta(k) = \max\{\Delta(c), \Delta_1(k)\}$.

Thus, we have proved, that for any $k \geqslant 1$ there exist $\Delta(k)$, so that for all $\Delta \geqslant \Delta(k)$ and $\delta \geqslant \lfloor \Delta/k \rfloor$ there exist $x, y, p \in (0,1)$ satisfying (2) and (3). Hence, the assumptions of Lovász Local Lemma are satisfied and Lemma 1 guarantees the existence of a second hamiltonian cycle C'. The proof is complete. $\square$

3. Bounds on $\Delta(k)$

Given number $k \geqslant 1$. Denote by $\Delta(k)$ the smallest Δ so that there exist numbers $x, y, p \in (0,1)$ satisfying inequalities (2) and (3) for $\delta = \lfloor \Delta/k \rfloor$. In this section, we demonstrate how to calculate an upper bound on $\Delta(k)$ which is very close to its real value. Unfortunately, we are not able to give an upper bound on $\Delta(k)$ in form of a function with variables Δ, k.

We set $y = 1/(\Delta^2 - 5\Delta + 7)$, $p = \sqrt{x}(1-x)c$, where $c = (1-y)^{2\Delta-3}$ and x to be a point at which the function $f(x) = [1 - \sqrt{x}(1-x)c]/(1-x)^{2k}$ attains its minimum on $(0,1)$, hence $x = x_r$, $0 < x_r < 1$, where x_r is a root of the equation

$$(4k-3)x^2 - (4k-4)x + \frac{4k}{c}\sqrt{x} - 1 = 0.$$

Setting $c = (1-y)^{2\Delta-3}$ instead of $c = (1-y)^{2\Delta-2}$ guarantees strict inequality in (2). Then, find the smallest value of Δ' satisfying (3). A calculus exercise, left to the reader, shows that for all $\Delta \geqslant \Delta'$, (3) is satisfied, i.e., that Δ' is an upper bound on $\Delta(k)$.

To get a lower bound on $\Delta(k)$ we consider an inequality

$$p^2 \leqslant x(1-x)^2. \tag{5}$$

Clearly, (5) is implied by (2). Thus, the minimum Δ_m for which there exist $x, y, p \in (0,1)$ satisfying (5) and (3) for $\delta = \lfloor \Delta/k \rfloor$ provides a lower bound on $\Delta(k)$. To calculate Δ_m we set $y = 1/(\Delta^2 - 5\Delta + 7)$ as y does not occur in (5), and the maximum of the function $g(y) = y(1-y)^n$ is attained at $y = 1/(n+1)$. Further, for any choice of x, the optimal value for p is $p = \sqrt{x}(1-x)$, as (5) will be satisfied and the value of $(1-p)^{\Delta/k-1}$ will be the minimum possible for given x. Thus, the best choice for x is the value at which the function $f(x) = 1/((1-x)^{2k-2})((1-\sqrt{x}(1-x))/(1-x)^{2k})^{\Delta/k-1}$ attains its minimum, i.e., the root x_r, $0 < x_r < 1$, of the equation

$$(4k-4)\sqrt{x}(1-x)^{2k} + \left(\frac{\Delta}{k} - 1\right)(4k\sqrt{x} + (4k-3)x^2 - (4k-4)x - 1) = 0.$$

The smallest value of Δ satisfying (3) is the sought value Δ_m.

Using a computer to carry out the above calculations, one obtains

$$70 \leqslant \Delta(1) \leqslant 73, 90 \leqslant \Delta(1.1) \leqslant 93, 189 \leqslant \Delta(1.5) \leqslant 192, 378 \leqslant \Delta(2) \leqslant 382,$$

$$3295 \leqslant \Delta(5) \leqslant 3300, \Delta(7) = 5010, \Delta(50) = 545\,800.$$

The bounds on $\Delta(1)$ show that the approach of this paper, which is, for $k=1$, identical to the methods of Thomassen [7], cannot be used to guarantee a second hamiltonian cycle in a r-regular hamiltonian graph for $r \leqslant 69$. This contradicts a comment in [7] that 4-regular case is perhaps the only case that needs a special method.

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